CHAPTER ONE

Introduction

In this book, gravity waves on water and their interaction with oscillating systems (having zero forward speed) are approached from a somewhat interdisciplinary point of view. Before the matter is explored in depth, a comparison is briefly made between different types of waves, including acoustic waves and electromagnetic waves, drawing the reader's attention to some analogies and dissimilarities. Oscillating systems for generating or absorbing waves on water are analogues of loudspeakers or microphones in acoustics, respectively. In electromagnetics the analogues are transmitting or receiving antennae in radio engineering, and light-emitting or light-absorbing atoms in optics.

The discussion of waves is, in this book, almost exclusively limited to waves of sufficiently low amplitudes for linear analysis to be applicable. Several other books (see, e.g., the monographs by Mei,[1] Faltinsen,[2] Sarpkaya and Isaacson[3] or Chakrabarti[4]) treat the subject of large ocean waves and extreme wave loads, which are so important for determining the survival ability of ships, harbours and other ocean structures. In contrast, the purpose of this book is to convey a thorough understanding of the interaction between waves and oscillations, when the amplitudes are low, which is true most of the time. For example, on one hand, for a wave-power plant the income is determined by the annual energy production, which is essentially accrued during most times of the year, when amplitudes are low, that is, when linear interaction is applicable. On the other hand, as with many other types of ocean installations, wave-power plants also have their expenses, to a large extent, determined by the extreme-load design. The technological aspects related to conversion and useful application of wave energy are not covered in the present book. Readers interested in such subjects are referred to other literature.[5–7]

The content of the subsequent chapters is outlined below. At the end of each chapter, except the first, there is a collection of problems.

Chapter 2 gives a mathematical description of free and forced oscillations in the time domain as well as in the frequency domain. An important purpose is to introduce students to the very useful mathematical tool represented by the complex representation of sinusoidal oscillations. The mathematical connection between complex amplitudes and Fourier transforms is treated. Linear systems are discussed in a rather general way, and for a causal linear system in particular, the Kramers-Kronig relations are derived. A simple mechanical oscillating system is analysed to some extent, the concept of mechanical impedance is introduced and a discussion of energy accounting in the system is included to serve as a tool for physical explanation, in subsequent chapters, of the so-called hydrodynamic "added mass".

In Chapter 3 a brief comparison is made of waves on water with other types of waves, in particular with acoustic waves. The concepts of wave dispersion, phase velocity and group velocity are introduced. In addition the transport of energy associated with propagating waves is considered, and the radiated power from a radiation source (wave generator) is mathematically expressed in terms of a phenomenologically defined radiation resistance. The radiation impedance, which is a complex parameter, is also introduced in a phenomenological way. For mechanical waves (such as acoustic waves and waves on water) its imaginary part may be represented by an added mass. Finally in Chapter 3, an analysis is given of the absorption of energy from a mechanical wave by means of a mechanical oscillation system of the simple type considered in Chapter 2. The optimum parameters of this system for maximising the absorbed energy are discussed. The maximum is obtained at resonance.

From Chapter 4 onward, a deeper hydrodynamic discussion of water waves is the main subject. With an assumption of inviscid and incompressible fluid and irrotational fluid motion, the hydrodynamic potential theory is developed. With the linearisation of fluid equations and boundary conditions, the basic equations for low-amplitude waves are derived. In most of the following discussions, either infinite water depth or finite, but constant, water depth is assumed. Dispersion and wave-propagation velocities are studied, and plane and circular waves are discussed in some detail. Also non-propagating, evanescent plane waves are considered. Another studied subject is wave-transported energy and momentum. The spectrum of real sea waves is treated only briefly in the present book. The rather theoretical Sections 4.7 and 4.8, which make extensive use of Green's theorem, may be omitted at the first reading, and then be referred to as needed during the study of the remaining chapters of the book. Whereas most of Chapter 4 is concerned with discussions in the frequency domain, the last section contains discussions in the time domain.

The subject of Chapter 5 is interactions between waves and oscillating bodies, including wave generation by oscillating bodies as well as forces induced by waves on the bodies. Initially six-dimensional generalised vectors are introduced which correspond to the six degrees of freedom for the motion of an immersed (three-dimensional) body. The radiation impedance, known from the phenomenological introduction in Chapter 3, is now defined in a hydrodynamic formulation, and, for a three-dimensional body, extended to a 6×6 matrix. In a later part of the chapter the radiation impedance matrix is extended to the case of a finite number of interacting, radiating, immersed bodies. For this case the generalised excitation force vector is decomposed into two parts, the Froude-Krylov part and the diffraction part, which are particularly discussed in the "small-body" (or "long-wavelength") approximation. From Green's theorem (as mentioned in the summary of Chapter 4) several useful reciprocity theorems are derived, which relate excitation force and radiation resistance to each other or to "far-field coefficients" (or "Kochin functions"). Subsequently these theorems are applied to oscillating systems consisting of concentric axisymmetric bodies or of two-dimensional bodies. The occurrence of singular radiation-resistance matrices is discussed in this connection. Whereas most of Chapter 5 is concerned with discussions in the frequency domain, two sections, Sections 5.3 and 5.9, contain discussions in the time domain. In the latter section motion response is the main subject. In the former section two hydrodynamic impulse-response functions are considered; one of them is causal and, hence, has to obey the Kramers-Kronig relations.

The extraction of wave energy is the subject of Chapter 6, which starts by explaining wave absorption as a wave-interference phenomenon. Toward the end of the chapter (Section 6.4) a study is made of the absorption of wave energy by means of a finite number of bodies oscillating in several (up to six) degrees of freedom. This discussion provides a physical explanation of the quite frequently encountered cases of singular radiation-resistance matrices, as mentioned above (also see Sections 5.7 and 5.8). However, the central part of Chapter 6 is concerned with wave-energy conversion which utilises only a single body oscillating in just one degree of freedom. With the assumption that an external force is applied to the oscillating system, for the purpose of power takeoff and optimum control of the oscillation, this discussion has a different starting point than that given in the last part of Chapter 3. The conditions for maximising the converted power are also studied for the case in which the body oscillation has to be restricted as a result of its designed amplitude limit or because of the installed capacity of the energy-conversion machinery.

Oscillating water columns (OWCs) are mentioned briefly in Chapter 4 and considered in greater detail in Chapter 7, where their interaction with incident waves and radiated waves is the main subject of study. Two kinds of interaction are considered: the radiation problem and the excitation problem. The radiation problem concerns the radiation of waves which is due to an oscillating dynamic air pressure above the internal air-water interfaces of the OWCs. The excitation problem concerns the oscillation which is due to an incident wave when the dynamic air pressure is zero for all OWCs. Comparisons are made with corresponding wave-body interactions. Also, wave-energy extraction by OWCs is discussed. Finally in this chapter, the case is considered in which several OWCs and several oscillating bodies are interacting with waves.

CHAPTER TWO

Mathematical Description of Oscillations

In this chapter, which is a brief introduction to the theory of oscillations, a simple mechanical oscillation system is used to introduce concepts such as free and forced oscillations, state-space analysis and representation of sinusoidally varying physical quantities by their complex amplitudes. In order to be somewhat more general, causal and non-causal linear systems are also looked at and Fourier transform is used to relate the system's transfer function to its impulse response function. With an assumption of sinusoidal (or "harmonic") oscillations, some important relations are derived which involve power and stored energy on one hand, and the parameters of the oscillating system on the other hand. The concepts of resonance and bandwidth are also introduced.

2.1 Free and Forced Oscillations of a Simple Oscillator

Let us consider a simple mechanical oscillator in the form of a mass-spring-damper system. A mass m is suspended through a spring S and a mechanical damper R, as indicated in Figure 2.1. Because of the application of an external force F the mass has a position displacement x from its equilibrium position.

Newton's law gives

$$m\ddot{x} = F + F_R + F_S, \tag{2.1}$$

where the spring force and the damper force are $F_S = -Sx$ and $F_R = -R\dot{x}$, respectively.

If we assume that the spring and the damper have linear characteristics, then the "stiffness" S and the "mechanical resistance" R are coefficients of proportionality, independent of the displacement x and of the velocity $u = \dot{x}$. Then Newton's law gives the following linear differential equation with constant coefficients,

$$m\ddot{x} + R\dot{x} + Sx = F, \tag{2.2}$$

where an overdot is used to denote differentiation with respect to time t.

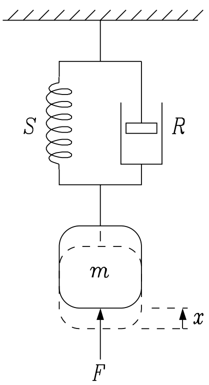

Figure 2.1: Mechanical oscillator composed of a mass-spring-damper system.

2.1.1 Free Oscillation

If the external force is absent, that is, $F = 0$, we may have so-called free oscillation if the system is released at a certain instant $t = 0$, with some initial energy

$$W_0 = W_{p0} + W_{k0} = Sx_0^2/2 + mu_0^2/2, \tag{2.3}$$

written here as a sum of potential and kinetic energy, where x_0 is the initial displacement and u_0 the initial velocity. It is easy to show (see Problem 2.1) that the general solution to Eq. (2.2), when $F = 0$, is

$$x = (C_1 \cos \omega_d t + C_2 \sin \omega_d t)\, e^{-\delta t}, \tag{2.4}$$

where

$$\delta = R/2m, \quad \omega_0 = \sqrt{S/m}, \quad \omega_d = \sqrt{\omega_0^2 - \delta^2} \tag{2.5}$$

are the damping coefficient, the undamped natural angular frequency and the damped angular frequency, respectively. The integration constants C_1 and C_2 may be determined from the initial conditions as (see Problem 2.1)

$$C_1 = x_0, \quad C_2 = (u_0 + x_0\delta)/\omega_d. \tag{2.6}$$

For the particular case of zero damping force, the oscillation is purely sinusoidal with a period $2\pi/\omega_0$, which is the so-called natural period of the oscillator. The free oscillation as given by Eq. (2.4) is an exponentially damped sinusoidal oscillation with "period" $2\pi/\omega_d$, during which a fraction $1 - \exp(-4\pi\delta/\omega_d)$ of the energy in the system is lost, as a result of power consumption in the damping resistance R. We define the oscillator's *quality factor* Q as the ratio between the stored energy and the average energy loss during a time interval of length $1/\omega_d$:

$$Q = \left(1 - e^{-2\delta/\omega_d}\right)^{-1}. \tag{2.7}$$

If the damping coefficient δ is small, then Q is large. When $\delta/\omega_0 \ll 1$, the following expansions (see Problem 2.2) may be useful:

$$Q = \frac{\omega_0}{2\delta}\left(1 + \frac{\delta}{\omega_0} - \frac{1}{6}\frac{\delta^2}{\omega_0^2} + \mathcal{O}\left\{\frac{\delta^3}{\omega_0^3}\right\}\right)$$
$$\approx \frac{\omega_0}{2\delta} = \frac{\omega_0 m}{R} = \frac{S}{\omega_0 R} = \frac{(Sm)^{1/2}}{R}, \tag{2.8}$$
$$\frac{\delta}{\omega_0} = \frac{1}{2Q}\left(1 + \frac{1}{2Q} + \frac{5}{24Q^2} + \mathcal{O}\{Q^{-3}\}\right) \approx \frac{1}{2Q}. \tag{2.9}$$

As a result of the energy loss, the freely oscillating system comes eventually to rest. The free oscillation is "overdamped" if ω_d is imaginary, that is, if $\delta > \omega_0$ or $R > 2(Sm)^{1/2}$. [The quality factor Q, as defined by Eq. (2.7), is then complex and it loses its physical significance.] Then the general solution of the differential equation (2.2) is a linear combination of two real, decaying exponential functions. The case of "critical damping", that is, when $R = 2(Sm)^{1/2}$ or $\omega_d = 0$, requires special consideration, which we omit here. (See, however, Problem 2.11.)

2.1.2 Forced Oscillation

When the differential equation (2.2) is inhomogeneous, that is, if $F = F(t) \neq 0$, the general solution may be written as a particular solution plus the general solution (2.4) of the corresponding homogeneous equation (corresponding to $F = 0$). Let us now consider the case in which the driving external force $F(t)$ has a sinusoidal time variation with angular frequency $\omega = 2\pi/T$, where T is the period. Let

$$F(t) = F_0 \cos(\omega t + \varphi_F), \tag{2.10}$$

where F_0 is the amplitude and φ_F the phase constant for the force. It is convenient to choose a particular solution of the form where

$$x(t) = x_0 \cos(\omega t + \varphi_x) \tag{2.11}$$

is the position, and

$$u(t) = \dot{x}(t) = u_0 \cos(\omega t + \varphi_u) \tag{2.12}$$

is the corresponding velocity of the mass m. Here the amplitudes are related by $u_0 = \omega x_0$ and the phase constants by $\varphi_u - \varphi_x = \pi/2$. For Eq. (2.11) to be a particular solution of the differential equation (2.2), it is necessary that (see Problem 2.3) the excursion amplitude is

$$x_0 = \frac{u_0}{\omega} = \frac{F_0}{|Z|\omega} \tag{2.13}$$

and that the phase difference

$$\varphi = \varphi_F - \varphi_u = \varphi_F - \varphi_x - \pi/2 \tag{2.14}$$

is an angle which is in quadrant no. 1 or no. 4, and which satisfies

$$\tan\varphi = (\omega m - S/\omega)/R. \tag{2.15}$$

Here

$$|Z| = \{R^2 + (\omega m - S/\omega)^2\}^{1/2} \tag{2.16}$$

is the absolute value (modulus) of the complex mechanical impedance, which is discussed later.

The "forced oscillation", Eq. (2.11) or (2.12), is a response to the driving force, Eq. (2.10). Let us now assume that F_0 is independent of ω, and then discuss the responses $x_0(\omega)$ and $u_0(\omega)$, starting with $u_0(\omega) = F_0/|Z(\omega)|$. Noting that $|Z|_{\min} = R$ for $\omega = \omega_0 = (S/m)^{1/2}$ and that $|Z| \to \infty$ for $\omega = 0$ as well as for $\omega \to \infty$, we see that $(u_0/F_0)_{\max} = 1/R$ for $\omega = \omega_0$ and that $u_0(0) = u_0(\infty) = 0$. We have resonance at $\omega = \omega_0$, where the "reactive" contribution $\omega m - S/\omega$ to the mechanical impedance vanishes. Graphs of the non-dimensionalised velocity response $\sqrt{Sm}\,u_0/F_0$ versus ω/ω_0 are shown in Figure 2.2 for $\sqrt{Sm}/R$ equal to 10 and 0.5. Note that the graphs are symmetric with respect to $\omega = \omega_0$ when the frequency scale is logarithmic. The

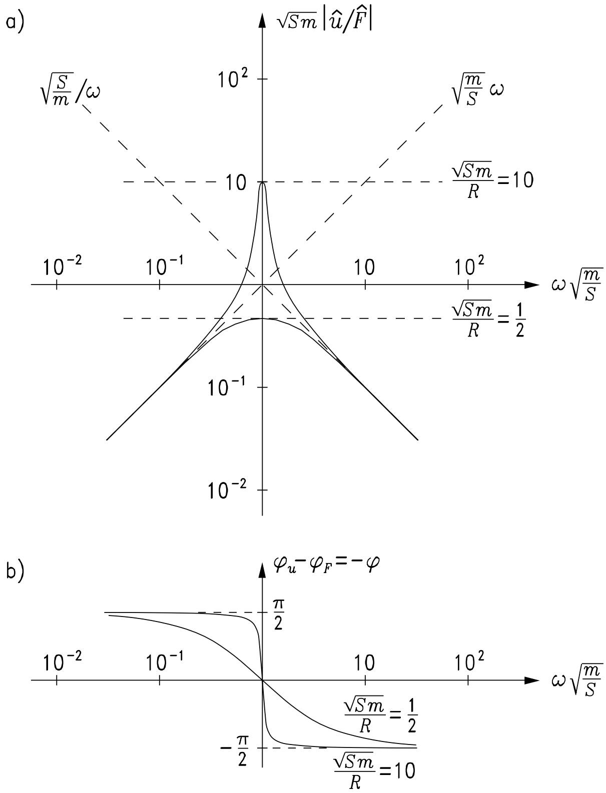

Figure 2.2: Frequency response of relation between velocity u and applied force F in normalised units, for two different values of the damping coefficient. a) Amplitude (modulus) response with both scales logarithmic. b) Phase response with linear scale for the phase difference.

phase difference φ as given by Eq. (2.15) is also shown in Figure 2.2. The graphs of Figure 2.2, where the amplitude response is presented in a double logarithmic diagram and the phase response in a semilogarithmic diagram, are usually called Bode plots or Bode diagrams.[8] Next, we consider the resonance bandwidth, that is, the frequency interval $(\Delta\omega)_{\text{res}}$, where

$$\frac{u_0(\omega)}{F_0} > \frac{1}{\sqrt{2}}\left(\frac{u_0}{F_0}\right)_{\max} = \frac{1}{R\sqrt{2}}, \tag{2.17}$$

that is, where the kinetic energy exceeds half of the maximum value. At the upper and lower edges of the interval, ω_u and ω_l, the two terms of the radicand in Eq. (2.16) are equally large. Thus, we have

$$\omega_u m - S/\omega_u = R = S/\omega_l - \omega_l m. \tag{2.18}$$

Instead of solving these two equations, we note from the above-mentioned symmetry that $\omega_u\omega_l = \omega_0^2 = S/m$, that is, $S/\omega_u = m\omega_l$ and $S/\omega_l = m\omega_u$. Evidently

$$(\Delta\omega)_{\text{res}} = \omega_u - \omega_l = R/m = 2\delta. \tag{2.19}$$

The relative bandwidth is

$$\frac{(\Delta\omega)_{\text{res}}}{\omega_0} = \frac{2\delta}{\omega_0} = \frac{R}{\sqrt{Sm}}, \tag{2.20}$$

and it is seen that this is inverse to the maximum non-dimensionalised velocity response $\sqrt{Sm}\,u_0(\omega_0)/F_0 = \sqrt{Sm}/R$ as indicated on the graph in Figure 2.2. From Eq. (2.8) we see that this is approximately equal to the quality factor Q, when this is large ($Q \gg 1$). In the same case,

$$(\Delta\omega)_{\text{res}}/\omega_0 \approx 1/Q. \tag{2.21}$$

Next we consider the excursion response, which, in non-dimensionalised form, may be written as $Sx_0(\omega)/F_0$. It equals unity for $\omega = 0$ and zero for $\omega = \infty$. At resonance its value is

$$Sx_0(\omega_0)/F_0 = S/\omega_0 R = (Sm)^{1/2}/R = \omega_0/2\delta, \tag{2.22}$$

as obtained by using Eqs. (2.13) and (2.16). Note that $x_0(\omega)$ has its maximum at a frequency which is lower than the resonance frequency. It can be shown (see Problem 2.4) that, if $R < (2Sm)^{1/2}$ or $\delta < \omega_0/\sqrt{2}$, then

$$\frac{Sx_{0,\max}}{F_0} = \frac{\omega_0}{2\delta}\left(1 - \frac{\delta^2}{\omega_0^2}\right)^{-1/2} \quad \text{at} \quad \omega = \omega_0\left(1 - \frac{2\delta^2}{\omega_0^2}\right)^{1/2}, \tag{2.23}$$

and if $R > (2Sm)^{1/2}$ then

$$\frac{Sx_{0,\max}}{F_0} = \frac{S\,x_0(0)}{F_0} = 1. \tag{2.24}$$

For large values of Q there is only a small difference between $x_0(\omega_0)/F_0$ and

$x_{0,\max}/F_0$. Using Eq. (2.9) we find

$$\frac{Sx_0(\omega_0)}{F_0} = Q - \frac{1}{2} + \frac{1}{24Q} + \mathcal{O}\{Q^{-2}\} \tag{2.25}$$

and

$$\frac{Sx_{0,\max}}{F_0} = Q - \frac{1}{2} + \frac{1}{6Q} + \mathcal{O}\{Q^{-2}\} \tag{2.26}$$

for

$$\omega = \omega_0\left(1 - \frac{1}{4Q^2} + \mathcal{O}\{Q^{-3}\}\right). \tag{2.27}$$

2.1.3 Electric Analogue: Remarks on the Quality Factor

For readers with a background in electric circuits, it may be of interest to note that the mechanical system of Figure 2.1 is analogous to the electric circuit shown in Figure 2.3, where an inductance m, a capacitance $1/S$ and an electric resistance R are connected in series. The force F is analogous to the driving voltage, the position x is analogous to the electric charge on the capacitance and the velocity u is analogous to the electric current. If Kirchhoff's law is applied to the circuit, Eq. (2.2) results. The "capacitive reactance" S/ω is related to the capacitance's ability to store electric energy (analogous to potential energy in the spring S of Figure 2.1), and the "inductive reactance" ωm is related to the inductance's ability to store magnetic energy (analogous to kinetic energy in the mass of Figure 2.1). The electric (or potential) energy is zero when $x(t) = 0$, and the magnetic (or kinetic) energy is zero when $u(t) = \dot{x}(t) = 0$. The instants for $x(t) = 0$ and those for $u(t) = 0$ are displaced by a quarter of a period $\pi/2\omega_0$, and at resonance the maximum values for the electric and magnetic (or potential and kinetic) energies are equal, $mu_0^2/2 = m\omega_0^2 x_0^2/2 = Sx_0^2/2$, because $m\omega_0 = S/\omega_0$. Thus, at resonance the stored energy is swinging back and forth between the two energy stores, twice every period of the system's forced oscillation.

By Eq. (2.7) we have defined the quality factor Q as the ratio between the stored energy and the average energy loss during a time interval $1/\omega_d$ of the free oscillation. An alternative definition would have resulted if instead the forced oscillation at resonance had been considered, for a time interval $1/\omega_0$ (and not $1/\omega_d$). The stored energy is $u_0^2/2$ and the average lost energy is $Ru_0^2/2\omega_0$ during a time $1/\omega_0$ (as is shown in more detail later, in Section 2.3). Such an alternative quality factor equals the right-hand side of the approximation (2.8), and

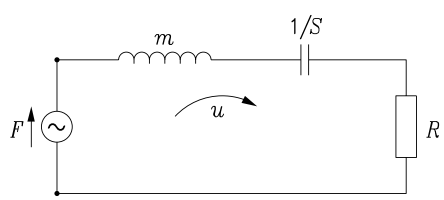

Figure 2.3: Electric analogue of the mechanical system shown in Figure 2.1.

would have been equal to the inverse of the relative bandwidth (2.20), to the non-dimensionalised excursion amplitude (2.22) at resonance (i.e., the ratio between the excursion at resonance and the excursion at zero frequency), and to the ratio of the reactance parts $\omega_0 m$ and S/ω_0 to the damping resistance R. The term quality factor is usually used only when it is large, $Q \gg 1$. In that case the relative difference between the two definitions is of little importance.

2.2 Complex Representation of Harmonic Oscillations

2.2.1 Complex Amplitudes and Phasors

When sinusoidal oscillations are dealt with, it is mathematically convenient to apply the method of complex representation, involving complex amplitudes and phasors. A great advantage with the method is that differentiation with respect to time is simply represented by multiplying with $i\omega$, where i is the imaginary unit ($i = \sqrt{-1}$). We consider again the forced oscillations represented by the excursion response $x(t)$ or velocity response $u(t)$ that is due to an applied external sinusoidal force $F(t)$, as given by Eqs. (2.11), (2.12) and (2.10), respectively.

With the use of Euler's formulas

$$e^{i\psi} = \cos\psi + i\sin\psi, \tag{2.28}$$

or, equivalently,

$$\cos\psi = (e^{i\psi} + e^{-i\psi})/2, \quad \sin\psi = (e^{i\psi} - e^{-i\psi})/2i, \tag{2.29}$$

the oscillating quantity $x(t)$ may be rewritten as

$$\begin{aligned} x(t) &= x_0\cos(\omega t + \varphi_x) \\ &= \frac{x_0}{2}e^{i(\omega t + \varphi_x)} + \frac{x_0}{2}e^{-i(\omega t + \varphi_x)} \\ &= \frac{x_0}{2}e^{i\varphi_x}e^{i\omega t} + \frac{x_0}{2}e^{-i\varphi_x}e^{-i\omega t}. \end{aligned} \tag{2.30}$$

Introducing the *complex amplitude* (see Figure 2.4),

$$\hat{x} = x_0 e^{i\varphi_x} = x_0\cos\varphi_x + i x_0 \sin\varphi_x, \tag{2.31}$$

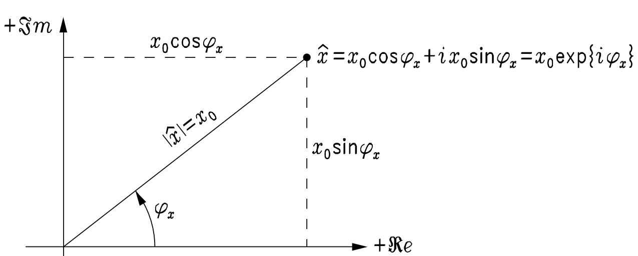

Figure 2.4: Complex-plane decomposition of the complex amplitude $\hat{x}$.