Motion and Expansion of a Viscous Vortex Ring: Elliptical Slowing Down and Diffusive Expansion

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1 Introduction

The motion of a vortex ring is a venerable problem, and, since the attempts of Helmholtz and Kelvin in the last century, extensive study has been made on various dynamical aspects, such as formation, traveling speed, waves, instability, interactions and so on. Concerning the steady solution for inviscid dynamics, analytical technique has been matured enough to make a highly nonlinear regime tractable. In contrast, the effect of viscosity on the nonlinear dynamics is poorly understood even for an isolated vortex ring.

In this article, we present a large-Reynolds-number asymptotic theory of the Navier–Stokes equations for the motion of an axisymmetric vortex ring of small cross-section. Our intention is to make the nonlinear effect amenable to analysis by constructing a framework for calculating higher-order asymptotics. The nonlinearity is featured by deformation of the core cross-section. We build a general formula for the translation speed incorporating the slowing-down effect caused by the elliptical deformation of the core. Moreover we show that viscosity has the action of expanding the ring radius, simultaneously with swelling the core; starting from an infinitely thin circular loop of radius R_0, the radii $R_s(t)$ of the loop of stagnation points relative to a comoving frame, $R_p(t)$ of the loop of peak vorticity, $R_c(t)$ of the centroid of vorticity all grow linearly in time t as $R_s \approx R_0 + 2.5902739\nu t/R_0$, $R_p \approx R_0 + 4.5902739\nu t/R_0$, and $R_c \approx R_0 + 3\nu t/R_0$. It is pointed out that the asymptotic values of R_p and R_c exhibit a discrepancy, at a finite Reynolds number, from the numerical result of Wang, Chu & Chang (1994).

To begin with, we briefly survey known results. Dyson (1893) (see also Fraenkel 1972) extended Kelvin's formula for the speed U of a thin axisymmetric vortex ring, steadily translating in an inviscid incompressible fluid of infinite extent, to third (virtually fourth) order in a small parameter $\varepsilon = \sigma/R_0$, the ratio of core radius σ to the ring radius R_0, as

$$U = \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8}{\varepsilon}\right) - \frac{1}{4} - \frac{3\varepsilon^2}{8}\left[\log\left(\frac{8}{\varepsilon}\right) - \frac{5}{4}\right] + O(\varepsilon^4 \log\varepsilon)\right\}, \qquad (1.1)$$

where Γ is the circulation carried by the ring. The vorticity is assumed to be in proportion to distance from the axis of symmetry. We consider Kelvin's formula (the first two terms) as the first-order and the $O(\varepsilon^2)$-terms as the third. The local self-induced flow consists not only of a uniform flow but also of a straining field. The latter manifests itself at $O(\varepsilon^2)$ and deforms the core into an ellipse, elongated in the propagating direction:

$$r = \sigma \left\{ 1 - \frac{3\varepsilon^2}{8} \left[\log\left(\frac{8}{\varepsilon}\right) - \frac{17}{12} \right] \cos 2\theta + \cdots \right\}, \tag{1.2}$$

where (r, θ) are local moving cylindrical coordinates about the core center which will be introduced in §2. The inclusion of the third-order term in the propagating velocity gives a remarkable improvement in approximation; (1.1) compares well even with the exact value for the 'fat' limit of Hill's spherical vortex (Fraenkel 1972). In this limit, the parameter ε is as large as $\sqrt{2}$ under a suitable normalisation. This surprising agreement encourages us to explore a higher-order approximation in more general circumstances.

Viscosity acts to diffuse vorticity, and the motion ceases to be steady. Its influence on the traveling speed, at large Reynolds number, was first addressed by Tung & Ting (1967), using the matched asymptotic expansions, for the case where the the vorticity is, at a virtual instant $t = 0$, a 'δ-function' concentrated on a circle of radius R_0. By a different method, Saffman (1970) succeeded in deriving an explicit formula, valid up to first order in $\epsilon \equiv (\nu/\Gamma)^{1/2}$, as

$$U = \frac{\Gamma}{4\pi R_0} \left[\log\left(\frac{8R_0}{2\sqrt{\nu t}}\right) - \frac{1}{2}(1 - \gamma + \log 2) + \cdots \right], \tag{1.3}$$

where ν is the viscosity, t is the time, and $\gamma = 0.57721566\cdots$ is Euler's constant (see also Callegari & Ting 1978). Wang, Chu & Chang (1994) employed a similar method to Tung & Ting (1967), but with a different choice $\sqrt{t}$ as small parameter, and gained a correction to (1.3) originating from the viscous diffusive effect. This correction vanishes in the limit of $\nu \to 0$. Unfortunately, the existing asymptotic theories all assume a circular symmetric core with a Gaussian distribution of vorticity. It implies that our knowledge of the nonlinear effect is restricted to $O(\epsilon)$. For comprehensive lists of theories of vortex rings, the article of Shariff & Leonard (1992) should be referred to.

Motivated by intriguing pattern variation of the dissipation field visualised from numerical data of simulations of fully developed turbulence, Moffatt, Kida & Ohkitani (1994) developed a large-Reynolds-number asymptotic theory for a steady stretched vortex tube subjected to uniform non-axisymmetric irrotational strain. They demonstrated that the higher-order asymptotics satisfactorily account for the fine structure of the dissipation field previously obtained by numerical computation (Kida & Ohkitani 1992). The corresponding planar problem, though unsteady, is dealt with in a similar manner, and an extension of the result of Ting & Tung (1965) to a higher order was achieved by Jiménez, Moffatt & Vasco (1996). The structure of the solutions have much in common; at leading order, a columnar vortex with circular cores, an exact solution of the Navier–Stokes equations, is obtained. A quadrupole component enters at $O(\nu/\Gamma)$, which is realised as the deformation of the core cross-section into an ellipse. The distinguishing feature is that the major axis of the ellipse is aligned at 45° to the principal axis of the external stain. This result leads us to expectat that the strained cross-section of a vortex ring, observed in nature, is established as an equilibrium between self-induced strain and viscous diffusion. Along the line of this scenario, we elucidate the structure of this strained core and its influence on the traveling speed of an axisymmetric vortex ring.

A powerful technique for our purpose is the method of matched asymptotic expansions. It has been previously developed to derive the velocity of a slender curved vortex tube (see, for example, Ting & Klein 1991). However this method is limited to $O(\epsilon^2)$ (Moore & Saffman 1972; Fukumoto & Miyazaki 1991). In the viscous case also, the self-induced strain, with the resulting elliptical deformation of the core, makes its appearance at $O(\epsilon^2)$, and its influences on the translation speed come up at $O(\epsilon^3)$. We are thus requested to extend asymptotic expansions to a higher order.

In §2, we state the general problem. The existing asymptotic formula for the potential flow associated with a circular vortex loop is not sufficient to carry through our program. In order to work out the correct inner limit of the outer solution, we devise, in §3, a technique to produce a systematic asymptotic expression of the Biot–Savart integral accommodating an arbitrary vorticity distribution. In §4, the inner expansions are scrutinised to $O(\epsilon)$ and are extended to $O(\epsilon^2)$. Based on these, we demonstrate in §5.1 that the radii of the loops of the stagnation points, maximum vorticity and vorticity centroid all grow linearly in time owing to the action of viscosity. Thereafter, we establish in §5.2 a general formula for the translating velocity of a vortex ring. In §6, an equation governing the temporal evolution of the axisymmetric vorticity at $O(\epsilon^2)$ is derived, and an integral representation of the exact solution is given, by which the formula of the preceding section can be closed.

A few ambiguous steps lying in previous theories stand as obstacles to proceeding to higher orders. These highlight the significance of the dipoles distributed along the core centerline and oriented in the propagating direction. It turns out that their strength needs to be prescribed at an initial instant, which solves the problem of undetermined constants at $O(\epsilon)$. As a by-product, a clear interpretation is provided of the general mechanism of the self-induced motion of a curved vortex tube. Because of the limitation of space, we must omit the technical details. A comprehensive account of our theory will be available in the paper of Fukumoto & Moffatt (2000).

2 Formulation

Consider an axisymmetric vortex ring of circulation Γ moving in an infinite expanse of viscous fluid with kinematic viscosity ν. We suppose that the circulation Reynolds number Re_Γ is very large:

$$Re_\Gamma = \Gamma/\nu \gg 1\,. \tag{2.1}$$

Two length scales are available, namely, measures of the core radius σ and the ring radius R_0. Suppose that their ratio σ/R_0 is very small. We focus attention on the translational motion of a 'quasi-steady' core. This means that we exclude stable or unstable wavy motion and fast core-area waves. Then, according to (1.1), the time-scale under question is of order $R_0/(\Gamma/R_0) = R_0^2/\Gamma$. The core spreads over this time to be of order $\sigma \sim (\nu t)^{1/2} \sim (\nu/\Gamma)^{1/2} R_0$. Our assumption of slenderness requires that the relevant small parameter $\epsilon(\ll 1)$ is

$$\epsilon = \sqrt{\nu/\Gamma}\,. \tag{2.2}$$

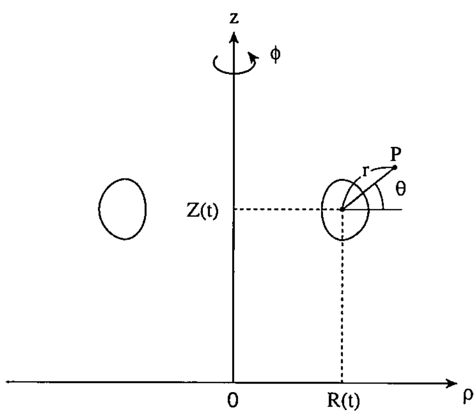

Figure 1

Choose cylindrical coordinates (ρ, ϕ, z) with the z-axis along the axis of symmetry and ϕ along the vortex lines as shown in Figure 1. We consider an axisymmetric distribution of vorticity $\boldsymbol{\omega} = \zeta(\rho, z)\boldsymbol{e}_\phi$ localised about the circle

$(\rho, z) = (R(t), Z(t))$, where $\boldsymbol{e}_\phi$ is the unit vector in the azimuthal direction. The Stokes streamfunction ψ is given by

$$\psi(\rho, z) = -\frac{\rho}{4\pi}\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\frac{\zeta(\rho', z')\rho'\cos\phi' d\rho' d\phi' dz'}{\sqrt{\rho^2 - 2\rho\rho'\cos\phi' + \rho'^2 + (z-z')^2}}. \tag{2.3}$$

The theorems of Kelvin and Helmholtz imply that determination of the ring motion necessitates a knowledge of the flow velocity in the vicinity of the core.

As is well known, the irrotational flow-velocity calculated from (2.3) for an infinitely thin core increases without limit primarily in inverse proportion to the distance from the core. In addition, it entails a logarithmic infinity originating from the curvature effect. These singularities may be resolved by matching the outer flow to an inner vortical flow which decays rapidly as the core center is approached. Thus we are led to inner and outer expansions (Ting & Tung 1965; Tung & Ting 1967). The inner region consists of the core itself and the surrounding toroidal region with thickness of the order of the core radius σ. There we develop an inner asymptotic expansion which matches at each level to the outer solution (2.3).

To this end, it is advantageous to introduce, in the axial plane, local polar coordinates (r, θ) moving with the core center[1] $(R(t), Z(t))$ with $\theta = 0$ in the ρ-direction (Figure 1):

$$\rho = R(t) + r\cos\theta, \qquad z = Z(t) + r\sin\theta. \tag{2.4}$$

Let us make the inner variables dimensionless. The radial coordinate is normalised by the core radius $\epsilon R_0 (= \sigma)$ and the local velocity $\boldsymbol{v} = (u, v)$, relative to the moving frame, by the maximum velocity $\Gamma/(\epsilon R_0)$. In view of (1.1), the normalisation parameter for the ring speed $(\dot{R}(t), \dot{Z}(t))$, the slow dynamics, should be Γ/R_0. The suitable dimensionless inner variables are thus defined as

$$\begin{gathered} r^* = r/\epsilon R_0, \quad t^* = t/\frac{R_0}{\Gamma}, \quad \psi^* = \frac{\psi}{\Gamma R_0}, \quad \zeta^* = \zeta/\frac{\Gamma}{R_0^2\epsilon^2}, \\ \boldsymbol{v}^* = \boldsymbol{v}/\frac{\Gamma}{R_0\epsilon}, \quad (\dot{R}^*, \dot{Z}^*) = (\dot{R}, \dot{Z})/\frac{\Gamma}{R_0}. \end{gathered} \tag{2.5}$$

The difference in normalisation between the last two of (2.5) should be kept in mind.

The equations handled in the inner region are the coupled system of the vorticity equation and the subsidiary relation between ζ and ψ. Dropping the asterisks, they take the following form:

$$\frac{\partial\zeta}{\partial t} + \frac{1}{\epsilon^2}\left(u\frac{\partial\zeta}{\partial r} + \frac{v}{r}\frac{\partial\zeta}{\partial\theta}\right) - \frac{1}{\epsilon\rho^2}\left(\frac{\partial\psi}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\cos\theta\right)\zeta$$

[1]The definition of the 'core center' will be discussed at some length in §4.2

$$= \hat{\nu}\left[\Delta\zeta + \frac{\epsilon}{\rho}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\zeta - \frac{\epsilon^2}{\rho^2}\zeta\right], \tag{2.6}$$

$$\zeta = \frac{1}{\rho}\Delta\psi - \frac{\epsilon}{\rho^2}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\psi, \tag{2.7}$$

where $\hat{\nu} = 1$, $\rho = R + \epsilon r\cos\theta$, and Δ is the two-dimensional Laplacian,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \tag{2.8}$$

and u and v are the r- and θ-components of the relative velocity $\boldsymbol{v}$:

$$u = \frac{1}{r\rho}\frac{\partial\psi}{\partial\theta} - \epsilon(\dot{Z}\sin\theta + \dot{R}\cos\theta), \tag{2.9a}$$

$$v = -\frac{1}{\rho}\frac{\partial\psi}{\partial r} - \epsilon(\dot{Z}\cos\theta - \dot{R}\sin\theta). \tag{2.9b}$$

We now postulate the following series expansions of the solution:

$$\zeta = \zeta^{(0)} + \epsilon\zeta^{(1)} + \epsilon^2\zeta^{(2)} + \epsilon^3\zeta^{(3)} + \cdots, \tag{2.10a}$$

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + \epsilon^3\psi^{(3)} + \cdots, \tag{2.10b}$$

$$R = R^{(0)} + \epsilon R^{(1)} + \epsilon^2 R^{(2)} + \cdots, \tag{2.10c}$$

$$Z = Z^{(0)} + \epsilon Z^{(1)} + \epsilon^2 Z^{(2)} + \cdots, \tag{2.10d}$$

where $\zeta^{(i)}$ and $\psi^{(i)}$ $(i = 0, 1, 2, 3, \cdots)$ are functions of r, θ and sometimes t. There arises $\log\epsilon$ as well, but we may conveniently take it to be of order unity, since multiples of $\log\epsilon$ happen to be ruled out at least to the above order. Inserting these expansions into (2.6) and (2.7), supplemented by (2.8)–(2.9b), and collecting terms with like powers of ϵ, we obtain the equations to be solved in the inner region.

The permissible solution must satisfy the condition:

$$u \text{ and } v \text{ are finite at } r = 0. \tag{2.11}$$

We emphasise that this condition is better than the restrictive one that $u = v = 0$ at $r = 0$. The requirement that it smoothly match the asymptotic form, valid in the vicinity of the core, of the outer solution will determine the values of $\dot{R}^{(i)}$ and $\dot{Z}^{(i)}$ $(i = 0, 1, 2, \cdots)$. This procedure was already performed by Tung & Ting (1967) and Callegari & Ting (1978) and others, up to first order. Our aim is to explore the second and third orders. Before that, we reconsider the earlier results.

3 Outer solution

For a circular vortex loop of unit strength placed at $(\rho, z) = (R, Z)$, $\zeta = \delta(\rho - R)\delta(z - Z)$ and the Stokes streamfunction (2.3) simplifies to

$$\psi_m(\rho, z; R) = -\frac{\rho}{4\pi}\int_0^{2\pi} \frac{R\cos\phi' d\phi'}{\sqrt{\rho^2 - 2\rho R\cos\phi' + R^2 + (z-Z)^2}}. \tag{3.1}$$

Use of the complete elliptic integrals K and E of the first and second kinds converts (3.1) into Maxwell's well-known formula. We call ψ_m the 'monopole field'. With the aid of the asymptotic behaviour of K and E for modulus close to unity, the asymptotic form of ψ_m for $r \ll R$ is obtainable at once (Dyson 1893; Tung & Ting 1967):

$$\begin{aligned} \psi_m = & -\frac{\Gamma R}{2\pi}\left\{\log\left(\frac{8R}{r}\right) + \frac{r}{2R}\left[\log\left(\frac{8R}{r}\right) - 1\right]\cos\theta \right. \\ & + \frac{r^2}{2^4 R^2}\left(\left[2\log\left(\frac{8R}{r}\right) + 1\right] + \left[-\log\left(\frac{8R}{r}\right) + 2\right]\cos 2\theta\right) \\ & \left. + \frac{r^3}{2^6 R^3}\left(\left[-3\log\left(\frac{8R}{r}\right) + 1\right]\cos\theta + \left[\log\left(\frac{8R}{r}\right) - \frac{7}{3}\right]\cos 3\theta\right)\right\} \\ & + \cdots . \end{aligned} \tag{3.2}$$

It turns out however that, when going to higher orders, (3.1) is not enough to qualify as the outer solution. Investigation of the detailed structure of (2.3) is unavoidable.

For this purpose, it is expedient to adapt Dyson's 'shift operator' technique to an arbitrary distribution of vorticity, and to cast (2.3) in the following form:

$$\psi = -\frac{\rho}{4\pi}\iint_{-\infty}^{\infty} dx' d\hat{z}' \zeta(x', \hat{z}') e^{x'\frac{\partial}{\partial R} - \hat{z}'\frac{\partial}{\partial \hat{z}}} \int_0^{2\pi} \frac{R\cos\phi' d\phi'}{\sqrt{\rho^2 - 2\rho R\cos\phi' + R^2 + \hat{z}^2}}, \tag{3.3}$$

where $(x, \hat{z}) = (\rho - R, z - Z)$ are local Cartesian coordinates attached to the moving frame, and ζ is rewritten in terms of them. Hereafter, we use z for $\hat{z}$. Supposing rapid decrease of vorticity with distance from the local origin $r = 0$, the exponential function of the operators is formally expanded in Taylor series as

$$\begin{aligned} \psi(\rho, z) = & \iint_{-\infty}^{\infty} dx' dz' \zeta(x', z')\left\{1 + \left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right) + \frac{1}{2!}\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right)^2 \right. \\ & + \frac{1}{3!}\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right)^3 + \frac{1}{4!}\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right)^4 + \frac{1}{5!}\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right)^5 \\ & \left. + \frac{1}{6!}\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right)^6 + \cdots\right\}\psi_m(\rho, z; R). \end{aligned} \tag{3.4}$$

We shall find in §4 that, up to $O(\epsilon^3)$, the vorticity distribution has the following dependence on the local polar coordinate θ:

$$\begin{aligned}\zeta(x,z) &= \zeta_0 + \epsilon\,\zeta_{11}^{(1)}\cos\theta + \epsilon^2(\zeta_0^{(2)} + \zeta_{21}^{(2)}\cos 2\theta)\\ &\quad + \epsilon^3(\zeta_{11}^{(3)}\cos\theta + \zeta_{12}^{(3)}\sin\theta + \zeta_{31}^{(3)}\cos 3\theta) + \cdots,\end{aligned} \tag{3.5}$$

where $\zeta_{ij}^{(k)}$ are functions of r and t, and k stands for the order of perturbation, i labels the Fourier mode with $j = 1$ and 2 corresponding to $\cos i\theta$ and $\sin i\theta$ respectively.

With this form, (3.2) and (3.5), along with (2.10c), are substituted into (3.4) and the resulting expression is made dimensionless by use of the normalization (2.5). Using, in advance, $R^{(1)} = 0$ and (6.5), we eventually arrive at the asymptotic development of the Biot–Savart law, valid to $O(\epsilon^3)$, in a region $r \ll R$ surrounding the core:

$$\begin{aligned}\psi = &-\frac{R^{(0)}\Gamma}{2\pi}\log\left(\frac{8R^{(0)}}{\epsilon r}\right) + \epsilon\left\{-\frac{\Gamma}{4\pi}\left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right) - 1\right] r\cos\theta + d^{(1)}\frac{\cos\theta}{r}\right\}\\
&+\epsilon^2\left\{-\frac{\Gamma}{2^5\pi R^{(0)}}\left(\left[2\log\left(\frac{8R^{(0)}}{\epsilon r}\right)+1\right]r^2 - \left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right) - 2\right] r^2\cos 2\theta\right)\right.\\
&\quad\left.+\frac{d^{(1)}}{2R^{(0)}}\left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right) + \frac{\cos 2\theta}{2}\right] - \frac{\Gamma R^{(2)}}{2\pi}\log\left(\frac{8R^{(0)}}{\epsilon r}\right) + q^{(2)}\frac{\cos 2\theta}{r^2}\right\}\\
&+\epsilon^3\left\{\frac{\Gamma}{2^7\pi (R^{(0)})^2}\left(\left[3\log\left(\frac{8R^{(0)}}{\epsilon r}\right) - 1\right] r^3\cos\theta - \left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right) - \frac{7}{3}\right] r^3\cos 3\theta\right)\right.\\
&\quad - \frac{d^{(1)}}{8(R^{(0)})^2}\left(\left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right) - \frac{7}{4}\right] r\cos\theta + \frac{r\cos 3\theta}{4}\right) - \frac{\Gamma R^{(2)}}{4\pi R^{(0)}} r\cos\theta\\
&\quad - \frac{1}{2\pi}\left(\frac{1}{4}\left[2\pi\int_0^\infty r^3\zeta_0^{(2)}dr\right] + R^{(0)}\left[\pi\int_0^\infty r^2\zeta_{11}^{(3)}dr\right] + \frac{1}{4}\left[\pi\int_0^\infty r^3\zeta_{21}^{(2)}dr\right]\right)\frac{\cos\theta}{r}\\
&\quad + \frac{q^{(2)}}{4R^{(0)}r}(\cos\theta + \cos 3\theta) - \frac{1}{\pi R^{(0)}}\left(\frac{1}{3\cdot 2^8}\left[2\pi\int_0^\infty r^7\zeta^{(0)}dr\right]\right.\\
&\quad - \frac{R^{(0)}}{8\cdot 4!}\left[\pi\int_0^\infty r^6\zeta_{11}^{(1)}dr\right] + \frac{(R^{(0)})^2}{4!}\left[\pi\int_0^\infty r^5\zeta_{21}^{(2)}dr\right]\\
&\quad\left.\left. + \frac{(R^{(0)})^3}{6}\left[\pi\int_0^\infty r^4\zeta_{31}^{(3)}dr\right]\right)\frac{\cos 3\theta}{r^3} - \frac{R^{(0)}}{2\pi}\left[\pi\int_0^\infty r^2\zeta_{12}^{(3)}dr\right]\frac{\sin\theta}{r}\right\}\\
&+\cdots,\end{aligned} \tag{3.6}$$

where

$$\Gamma = 2\pi\int_0^\infty r\zeta^{(0)}dr, \tag{3.7a}$$

($\Gamma = 1$ when dimensionless), and $d^{(1)}$ and $q^{(2)}$ are the strength of thedipole at $O(\epsilon)$ and quadrupole at $O(\epsilon^2)$:

$$d^{(1)} = -\frac{1}{2\pi}\left\{\frac{1}{4}\left[2\pi\int_0^\infty r^3\zeta^{(0)}dr\right] + R^{(0)}\left[\pi\int_0^\infty r^2\zeta_{11}^{(1)}dr\right]\right\}, \tag{3.7b}$$

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$$q^{(2)} = -\frac{1}{2\pi R^{(0)}}\left\{-\frac{1}{2^6}\left[2\pi\int_0^\infty r^5\zeta^{(0)}dr\right] + \frac{R^{(0)}}{8}\left[\pi\int_0^\infty r^4\zeta_{11}^{(1)}dr\right] + \frac{(R^{(0)})^2}{2}\left[\pi\int_0^\infty r^3\zeta_{21}^{(2)}dr\right]\right\}. \quad (3.7c)$$

The terms multiplied by Γ stem from $\Gamma\psi_m$, and only these have been previously employed as the outer solution. We now recognize that, at higher orders, the monopole field needs to be corrected by the induction velocity associated with the di-, quadru-, hexa-poles ... distributed along the centerline $r = 0$ of the core. In the light of (3.7b) and (3.7c), the detailed profile of vorticity in the core is necessary to evaluate these multi-pole induction terms. Parts of (3.6) supply the matching conditions on the inner solution. The distributions of $\zeta_{11}^{(1)}$, $\zeta_0^{(2)}$, $\zeta_{21}^{(2)}$, $\zeta_{11}^{(3)}$, $\zeta_{12}^{(3)}$ and $\zeta_{31}^{(3)}$ are as yet unknown, but will be determined successively by the inner expansions and the matching procedure.

4 Inner expansions up to second order

In this section, we recall the inner expansions at leading and first orders, developed by Tung & Ting (1967), Widnall, Bliss & Zalay (1971) and Callegari & Ting (1978), and extend them to second order.

4.1 Zeroth order

At $O(\epsilon^0)$, the Navier–Stokes equations reduce to the Jacobian form of the steady Euler equations:

$$[\zeta^{(0)}, \psi^{(0)}] \equiv \frac{1}{r}\frac{\partial(\zeta^{(0)}, \psi^{(0)})}{\partial(r,\theta)} = 0\,, \quad (4.1)$$

resulting in $\zeta^{(0)} = \mathcal{F}(\psi^{(0)})$, for some function $\mathcal{F}$.

Suppose that the flow $\psi^{(0)}$ has a single stagnation point at $r = 0$, the streamlines being all closed around that point. Then it is probable that the solution of (4.1), coupled with the ζ-ψ relation

$$\zeta^{(0)} = \frac{1}{R^{(0)}}\Delta\psi^{(0)}\,, \quad (4.2)$$

must be 'radial'; the streamlines are necessarily all circles (Moffatt *et al.* 1994). This statement may stand as a corollary of the theorem proved by Caffarelli & Friedman (1980) and Fraenkel (1999). In any event, we may certainly *assume* that $\psi^{(0)} = \psi^{(0)}(r)$.

The functional form of $\psi^{(0)}$ and $\zeta^{(0)}$ remains undetermined at this level of approximation, but is determined through the axisymmetric (or θ-averaged) part of the vorticity equation at $O(\epsilon^2)$:

$$\frac{\partial \zeta^{(0)}}{\partial t} = \left(\zeta^{(0)} + \frac{r}{2}\frac{\partial \zeta^{(0)}}{\partial r} \right) \frac{\dot{R}^{(0)}}{R^{(0)}} + \hat{\nu} \left(\frac{\partial^2 \zeta^{(0)}}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta^{(0)}}{\partial r} \right), \tag{4.3}$$

(Tung & Ting 1967). It follows that viscosity plays the role of selecting the distribution. For instance, we restrict our attention to a specific initial distribution of a 'δ-function' vorticity concentrated on the circle of radius $R^{(0)}$:

$$\zeta^{(0)} = \delta(\rho - R^{(0)})\delta(z - Z^{(0)}) \quad \text{at} \quad t = 0. \tag{4.4}$$

When $R^{(0)}$ is constant, to be shown in the next subsection, we obtain the Oseen diffusing vortex:

$$\zeta^{(0)} = \frac{1}{4\pi\hat{\nu}t} e^{-r^2/4\hat{\nu}t}. \tag{4.5}$$

In view of (2.9a), (2.9b) and (4.2), the leading-order variables are related to each other through

$$u^{(0)} = \frac{1}{R^{(0)}r}\frac{\partial \psi^{(0)}}{\partial \theta}, \quad v^{(0)} = -\frac{1}{R^{(0)}}\frac{\partial \psi^{(0)}}{\partial r}, \quad \zeta^{(0)} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r v^{(0)} \right). \tag{4.6}$$

These are integrated to provide $u^{(0)} = 0$ and, in the case of the Oseen vortex,

$$v^{(0)} = -\frac{1}{2\pi r}\left(1 - e^{-r^2/4\hat{\nu}t} \right), \quad \psi^{(0)} = \frac{R^{(0)}}{2\pi}\int_0^r \frac{1}{r'}\left(1 - e^{-r'^2/4\hat{\nu}t} \right) dr'. \tag{4.7}$$

This solution automatically fulfills the matching condition, the leading-order part of (3.6).

4.2 First order

Combining the vorticity equation with ζ-ψ relation at $O(\epsilon)$, we see that the first-order perturbation $\psi^{(1)}$ satisfies

$$(\Delta - a)\,\psi^{(1)} = -\cos\theta v^{(0)} + R^{(0)} r a(\dot{Z}^{(0)}\cos\theta - \dot{R}^{(0)}\sin\theta) + 2r\zeta^{(0)}\cos\theta, \tag{4.8}$$

where

$$a(r,t) = -\frac{1}{v^{(0)}}\frac{\partial \zeta^{(0)}}{\partial r}. \tag{4.9}$$

Here we have anticipated that $\zeta_0^{(1)} = 0$, which follows from an analysis of the vorticity equation at $O(\epsilon^3)$.

The solution satisfying the condition (2.11) at $O(\epsilon^3)$ is explicitly written in the following way. The θ-dependence is

$$\psi^{(1)} = \psi_{11}^{(1)}\cos\theta + \psi_{12}^{(1)}\sin\theta. \tag{4.10}$$