1
Properties of the S-matrix

In this chapter we specify the kinematics, define the normalisation of amplitudes and cross sections and establish the basic formalism used throughout. All mathematical functions used, and their properties, can be found in [9].

1.1 Kinematics

We consider first the two-body scattering process $1+2 \to 3+4$ of figure 1.1, where the particles have masses m_i and four-momenta P_i, $i = 1, \dots, 4$. Our notation is that the four-momentum of a particle is $P = (E, \mathbf{p})$, where E is its energy and $\mathbf{p}$ its three-momentum, and we write

$$P_1.P_2 = E_1E_2 - \mathbf{p}_1.\mathbf{p}_2. \tag{1.1}$$

The Lorentz-invariant variables s, t and u, called Mandelstam variables, are defined by

$$\begin{aligned} s &= (P_1 + P_2)^2 \\ t &= (P_1 - P_3)^2 \\ u &= (P_1 - P_4)^2 \end{aligned} \tag{1.2}$$

with the relation

$$s + t + u = \sum_{i=1}^{4} m_i^2\,. \tag{1.3}$$

Equation (1.3) means that a two-body amplitude is a function of only two independent variables. We shall normally take these to be s and t, with u defined via (1.3), and write the amplitude as $A(s,t)$. However, sometimes

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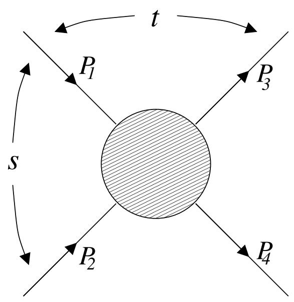

Figure 1.1. Two-body scattering process $1 + 2 \rightarrow 3 + 4$

it will be more appropriate to use s and u, or t and u, as the independent variables, and then write the amplitude as $A(s, u)$ or $A(t, u)$.

Figure 1.1 not only describes the scattering process $1 + 2 \rightarrow 3 + 4$ in the s-channel but, by reversing the signs of some of the four-momenta, it can also represent the t-channel process $1+\bar{3} \rightarrow \bar{2}+4$ and the u-channel process $1 + \bar{4} \rightarrow 3 + \bar{2}$, where the bar denotes the antiparticle.

In the s-channel centre-of-mass frame of the initial particles 1 and 2, the four-momenta are given explicitly by

$$\begin{aligned} P_1 &= (E_1, \mathbf{p}_1) \qquad & P_2 &= (E_2, -\mathbf{p}_1) \\ P_3 &= (E_3, \mathbf{p}_3) \qquad & P_4 &= (E_4, -\mathbf{p}_3) \end{aligned} \tag{1.4}$$

where E_i is the energy of particle i, $\mathbf{p}_1$ is the three-momentum of particle 1 and $\mathbf{p}_3$ the three-momentum of particle 3 in this frame. Then

$$s = (E_1 + E_2)^2 = (E_3 + E_4)^2 \tag{1.5}$$

and

$$\begin{aligned} E_1 &= \frac{1}{2\sqrt{s}}(s + m_1^2 - m_2^2) \qquad & E_2 &= \frac{1}{2\sqrt{s}}(s + m_2^2 - m_1^2) \\ E_3 &= \frac{1}{2\sqrt{s}}(s + m_3^2 - m_4^2) \qquad & E_4 &= \frac{1}{2\sqrt{s}}(s + m_4^2 - m_3^2) \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} \mathbf{p}_1^2 &= \frac{1}{4s}[s - (m_1 + m_2)^2]\,[s - (m_1 - m_2)^2] \\ \mathbf{p}_3^2 &= \frac{1}{4s}[s - (m_3 + m_4)^2]\,[s - (m_3 - m_4)^2]. \end{aligned} \tag{1.7}$$

From (1.2) and (1.4),

$$\begin{aligned} t &= m_1^2 + m_3^2 - 2(E_1E_3 - \mathbf{p}_1.\mathbf{p}_3) \\ &= m_1^2 + m_3^2 - 2(E_1E_3 - |\mathbf{p}_1||\mathbf{p}_3|\cos\theta_s) \\ u &= m_1^2 + m_4^2 - 2(E_1E_4 + \mathbf{p}_1.\mathbf{p}_3) \\ &= m_1^2 + m_4^2 - 2(E_1E_4 + |\mathbf{p}_1||\mathbf{p}_3|\cos\theta_s) \end{aligned} \tag{1.8}$$

where θ_s is the angle between the three-momenta of particles 1 and 3 in the s-channel centre-of-mass frame, that is it is the centre-of-mass-frame scattering angle.

The physical region for the s-channel is given by

$$s \geq (m_1 + m_2)^2 \qquad \text{and} \qquad -1 \leq \cos\theta_s \leq 1. \tag{1.9}$$

For arbitrary masses the boundary of the physical region as a function of s and t is rather complicated. It is simpler for equal masses $m_i = m$, $i = 1, \ldots, 4$, so that $\mathbf{p}_1 = \mathbf{p}_3 = \mathbf{p}$ and

$$\begin{aligned} s &= 4(\mathbf{p}^2 + m^2) \\ t &= -2\mathbf{p}^2(1 - \cos\theta_s) \\ u &= -2\mathbf{p}^2(1 + \cos\theta_s). \end{aligned} \tag{1.10}$$

The physical region for s-channel scattering is then given by $s \geq 4m^2$, $t \leq 0$ and $u \leq 0$. In this channel, s is an energy squared and each of t and u is a momentum transfer squared. Similarly the physical region for t-channel scattering is $t \geq 4m^2$, $u \leq 0$, $s \leq 0$; and for u-channel scattering it is $u \geq 4m^2$, $s \leq 0$, $t \leq 0$. The symmetry between s, t and u is readily demonstrated by plotting the physical regions in the s-t plane with the s and t axes inclined at 60°, as shown in figure 1.2.

1.2 The cross section

For orthonormal states $\langle f|$ and $|i\rangle$, that satisfy $\langle f|f\rangle = \langle i|i\rangle$ and $\langle f|f'\rangle = \delta_{ff'}$, the S-matrix element $\langle f|S|i\rangle$ is defined such that

$$P_{fi} = |\langle f|S|i\rangle|^2 = \langle i|S^\dagger|f\rangle\langle f|S|i\rangle \tag{1.11}$$

is the probability of $|f\rangle$ being the final state, given $|i\rangle$ as the initial state. If the set of orthonormal states $|f\rangle$ is complete,

$$\sum_f |f\rangle\langle f| = 1. \tag{1.12}$$

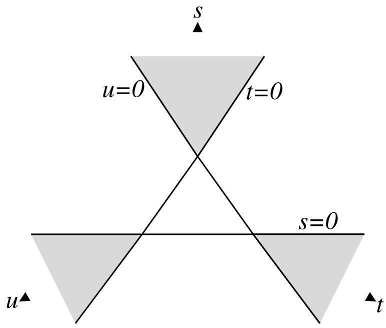

Figure 1.2. Physical regions for equal-mass scattering such as $\pi\pi \to \pi\pi$

Starting from the initial state $|i\rangle$, the probability of ending up in some final state must be unity so

$$1 = \sum_f |\langle f|S|i\rangle|^2 = \sum_f \langle i|S^\dagger|f\rangle\langle f|S|i\rangle = \langle i|S^\dagger S|i\rangle. \tag{1.13}$$

Since (1.13) must be true for any choice of the complete set of basis states $|i\rangle$ it follows that $S^\dagger S = 1$. Similarly the requirement that any final state $|f\rangle$ has originated from some initial state $|i\rangle$ yields $SS^\dagger = 1$. That is, S is unitary.

We now go over to the case of continuum states and specialise to a two-body initial state. The scattering matrix S is related to the transition matrix T by

$$\langle f|S|i\rangle = \langle P_1'P_2' \dots P_n'|S|P_1P_2\rangle = \delta_{fi} + i(2\pi)^4\delta^4(P^f - P^i)\,\langle f|T|i\rangle \tag{1.14}$$

where P^i is the sum of the initial four-momenta and P^f the sum of the final four-momenta. The scattering amplitude is normalised such that the transition rate per unit time per unit volume from the initial state $|i\rangle = |P_1P_2\rangle$ to the final state $|f\rangle = |P_1' \cdots P_n'\rangle$ is

$$R_{fi} = (2\pi)^4\delta^4(P^f - P^i)\,|\langle f|T|i\rangle|^2. \tag{1.15}$$

The total cross section for the reaction $12 \to n$ particles is

$$\sigma_{12\to n} = \frac{1}{4|\mathbf{p}_1|\sqrt{s}} \sum (2\pi)^4\delta^4(P^f - P^i)\,|\langle f_n|T|i\rangle|^2 \tag{1.16}$$

where the sum is over the momenta of the particles in the n-particle state $\langle f_n|$. That is, with $\delta^+(p^2 - m^2) = \delta(p^2 - m^2)\,\theta(p^0)$,

$$\sigma_{12\to n} = \frac{1}{4|\mathbf{p}_1|\sqrt{s}} \int \left(\prod_{i=1}^{n} \frac{d^4P_i'}{(2\pi)^4}\, 2\pi\delta^+(P_i'^2 - m_i^2) \right)$$
$$\times (2\pi)^4\delta^4\Big(\sum_{i=1}^{n} P_i' - P_1 - P_2\Big)\, |\langle P_1' \cdots P_n'|T|P_1P_2\rangle|^2$$
$$= \frac{1}{4|\mathbf{p}_1|\sqrt{s}} \int \left(\prod_{i=1}^{n} \frac{d^3p_i'}{2E_i(2\pi)^3} \right) (2\pi)^4\delta^4\Big(\sum_{i=1}^{n} P_i' - P_1 - P_2\Big)$$
$$\times |\langle P_1' \cdots P_n'|T|P_1P_2\rangle|^2. \tag{1.17}$$

Here, $\mathbf{p}_1$ is the initial momentum in the s-channel centre-of-mass frame. It is given by (1.7):

$$|\mathbf{p}_1|^2 s = (P_1.P_2)^2 - m_1^2m_2^2 = \tfrac{1}{4}[s-(m_1+m_2)^2]\,[s-(m_1-m_2)^2]. \tag{1.18}$$

We must use this in (1.17), which then gives the cross section in any frame: it is Lorentz invariant, and the momentum integrations may be performed in any frame.

We may calculate a differential cross section $d\sigma_{12\to n}/d\omega$. Typically, ω will be a momentum transfer between an initial and a final particle, or the corresponding scattering angle, or the energy of one of the final particles. To calculate the differential cross section, we first express ω as a function $\omega(P_i, P_f')$ of the various momenta, and then include $\delta(\omega - \omega(P_i, P_f'))$ in the integrations in (1.17). For example, when the final state contains just two particles and t is the momentum transfer defined in (1.2),

$$\frac{d\sigma_{12\to 34}}{dt} = \frac{1}{4|\mathbf{p}_1|\sqrt{s}} \int \frac{d^4P_3}{(2\pi)^4} 2\pi\delta^+({P_3}^2 - m_3^2)\, \frac{d^4P_4}{(2\pi)^4} 2\pi\delta^+({P_4}^2 - m_4^2)$$
$$\times (2\pi)^4\delta^4(P_1+P_2-P_3-P_4)|\langle P_3P_4|T|P_1P_2\rangle|^2\delta(t-(P_1-P_3)^2)$$
$$= \frac{1}{64\pi|\mathbf{p}_1|^2 s}|\langle P_3P_4|T|P_1P_2\rangle|^2\,\delta(t-(P_1-P_3)^2). \tag{1.19}$$

In the equal-mass case this gives

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s(s-4m^2)}\,|\langle P_3P_4|T|P_1P_2\rangle|^2. \tag{1.20}$$

The formulae in this section apply when the particles involved have no spin or, if they do have spin, when we average over initial spin states and sum over final spin states.

1.3 Unitarity and the optical theorem

Unitarity provides an important connection between the total cross section and the forward ($\theta_s = 0$) elastic scattering amplitude; this connection is known as the optical theorem. Because the operator S is unitary, so that $SS^\dagger = 1$, for any orthonormal states $\langle j|$ and $|i\rangle$

$$\delta_{ji} = \langle j|SS^\dagger)|i\rangle = \sum_f \langle j|S|f\rangle\langle f|S^\dagger)|i\rangle \tag{1.21}$$

where we have used the completeness relation (1.12). With the definition (1.14) of the T-matrix, this is

$$\langle j|T|i\rangle - \langle j|T^\dagger|i\rangle = (2\pi)^4 i \sum_f \delta^4(P^f - P^i)\langle j|T^\dagger|f\rangle\langle f|T|i\rangle. \tag{1.22}$$

For the particular case $j = i$,

$$2\,\mathrm{Im}\,\langle i|T|i\rangle = \sum_f (2\pi)^4 \delta^4(P^f - P^i)|\langle f|T|i\rangle|^2. \tag{1.23}$$

The right-hand side is (1.15) summed over f: it is the total transition rate. This gives us the total cross section, which is (1.17) summed over n, the number of final-state particles:

$$\sigma_{12}^{\mathrm{Tot}} = \frac{1}{2|\mathbf{p}_1|\sqrt{s}}\,\mathrm{Im}\,\langle i|T|i\rangle. \tag{1.24}$$

Here, $|\mathbf{p}_1|$ is again the magnitude of the initial centre-of-mass frame three-momentum, which is given by (1.18). $\langle i|T|i\rangle$ is the scattering amplitude for the reaction $1 + 2 \to 1 + 2$ with the direction of motion of the particles unchanged, that is it is the forward scattering amplitude, $\theta_s = 0$. For $m_3 = m_1$ and $m_4 = m_2$ the forward direction corresponds to $t = 0$. Then

$$\sigma_{12}^{\mathrm{Tot}} = \frac{1}{2|\mathbf{p}_1|\sqrt{s}}\,\mathrm{Im}\,A(s, t = 0) \tag{1.25}$$

where $A(s,t)$ is the elastic scattering amplitude. Equation (1.24) or (1.25) is the optical theorem.

1.4 Crossing and analyticity

The basic principle of crossing is that the same function $A(s,t)$ analytically continued to the three physical regions of figure 1.2 gives the corresponding scattering amplitude there, with s, t, u related by (1.3). This is obviously

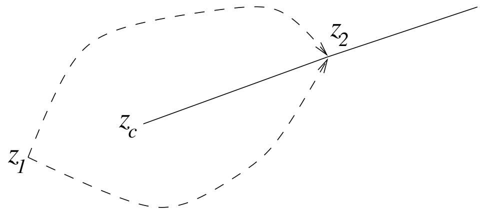

Figure 1.3. Paths of analytic continuation that pass round different sides of a branch point

true order by order for Feynman diagrams. For example Coulomb scattering ($e^-e^- \to e^-e^-$) and Bhabha scattering ($e^+e^- \to e^+e^-$) are described by the same Feynman diagrams.

It is necessary to make some assumption about the analytic structure of the scattering amplitude $A(s,t)$ in order to continue from one region to another. The assumption usually made is that any singularity has a dynamical origin. Poles are associated with bound states and thresholds give rise to cuts. For example in the s-plane a bound state of mass $m_B = \sqrt{s_B}$ will give rise to a pole at $s = s_B$ and there will be cuts with branch points corresponding to physical thresholds. These arise because of the unitarity condition (1.23). In this condition, $P^{f2} = s$ is the squared invariant mass of the state f, which shows that n-particle states contribute to the imaginary part of the amplitude if $\sqrt{s}$ is greater than the n-particle threshold energy. The threshold for producing a state in which the particles have masses $M_1, M_2, M_3, \dots$ is at $s = (M_1 + M_2 + M_3 + \cdots)^2$. In a model with only one type of particle, of mass m, the thresholds are at $s = 4m^2, 9m^2, \dots$. Each corresponds to a branch point of $A(s,t)$. When a function $f(z)$ of a complex variable z has a branch point at some point z_c, we attach a cut to the branch point, to remind us that continuing $f(z)$ from z_1 to z_2 along paths that pass to different sides of the branch point results in different values for the function: see figure 1.3. We say that $f(z)$ has a discontinuity across the cut. Since we may choose the point z_2 to lie in any direction relative to z_1, we must be prepared to draw the cut in any direction. It need not be a straight line. The only constraint is that one end of it is at $z = z_c$ and does not cross any other singularity. For $A(s,t)$, therefore, we need a cut attached to each branch point $s = 4m^2, 9m^2, 16m^2, \dots$. By convention, we draw each cut along the real axis, so that the one attached to $s = 4m^2$ passes through all the other branch points and effectively all these branch points need only one cut, the right-hand one in figure 1.4.

A consequence of the assumption of analyticity is crossing symmetry. Con-

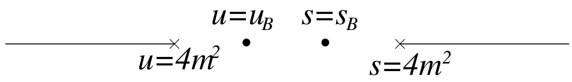

Figure 1.4. Poles and cuts in the complex s-plane for equal mass scattering for a given, fixed t. Recall that $u = 4m^2 - s - t$.

sider the scattering process

$$a + b \to c + d \tag{1.26}$$

and write its amplitude as $A_{a+b\to c+d}(s,t,u)$, reinstating the variable u for symmetry, but remembering that it is not independent being given in terms of s and t by (1.3). The physical region for the process (1.26) is $s > \max\{(m_a + m_b)^2, (m_c + m_d)^2\}$. In the equal-mass case, $t, u < 0$; in the unequal-mass case the constraint on t and u is more complicated, but most of the physical region lies in $t, u < 0$. The amplitude may be continued analytically to the region $t > \max\{(m_a + m_{\bar{c}})^2, (m_{\bar{b}} + m_d)^2\}$ and $s, u < 0$. This gives the amplitude for the t-channel process

$$a + \bar{c} \to \bar{b} + d \tag{1.27}$$

where $\bar{b}$ and $\bar{c}$ mean respectively the antiparticles of b and c. That is, we have

$$A_{a+\bar{c}\to\bar{b}+d}(t,s,u) = A_{a+b\to c+d}(s,t,u). \tag{1.28}$$

Similarly for the u-channel process

$$a + \bar{d} \to \bar{b} + d \tag{1.29}$$

we have

$$A_{a+\bar{d}\to\bar{b}+c}(u,t,s) = A_{a+b\to c+d}(s,t,u). \tag{1.30}$$

There are various mathematical results about the analytic properties of scattering amplitudes. Although these results are not complete, what is known is consistent with the assumption that the analytic structure in the complex s-plane for equal mass scattering is that shown in figure 1.4. The right-hand cut, from $s = 4m^2$ to ∞, arises from the physical thresholds in the s-channel. The pole at $s = s_B$ assumes that there is a bound state in the s-channel with mass $m_B = \sqrt{s_B}$. The left hand cut and pole arise respectively from the physical thresholds in the u-channel and an assumed u-channel bound state at $u = u_B$. The position of the singularities in the s-plane arising from u-channel effects is given by the relation (1.3). Thus the presence of a threshold at $u = u_0$ for positive u means that the amplitude $A(s,t)$ must have a cut along the negative real axis with a branch point at $s = \bar{s}_0 = 4m^2 - t - u_0$, so that $\bar{s}_0 = -t$ when $u_0 = 4m^2$. Equally, a bound-state pole at $u = u_B$ will give rise to a pole at $s = 4m^2 - t - u_B$. In figure 1.4 we have drawn the u-channel bound-state pole and the u-channel cut to the left of the corresponding s-channel singularities. However, they move as t varies and for physical values of t, $t \leq 0$, the u-channel pole is actually to the right of the s-channel pole, and when t is sufficiently large negative the two cuts actually overlap.

In perturbation theory, masses are assigned a small negative imaginary part, $m^2 \to m^2 - i\epsilon$, which is made to go to zero at the end of any calculation. The same $i\epsilon$ prescription is used outside the framework of perturbation theory; for example it makes Minkowski-space path integrals converge for large values of the fields. In figure 1.4, the $i\epsilon$ prescription pushes the branch point at $s = 4m^2$ downwards in the complex s-plane, and likewise the branch points corresponding to the higher thresholds, $s = 9m^2, s = 16m^2, \ldots$. As $\epsilon \to 0$, the branch points move back on to the real axis from below. That is, the physical s-channel amplitude is reached by analytic continuation down on to the real axis from the upper half of the complex s-plane. This is equivalent to saying that the physical amplitude is

$$\lim_{\epsilon \to 0} A(s + i\epsilon, t). \tag{1.31}$$

If we analytically continue it to real values of s between s_B and $4m^2$, there is no cut and the amplitude is real there[10]. The Schwarz reflection principle tells us that an analytic function $f(s)$ which is real for some range of real values of s satisfies

$$f(s^*) = [f(s)]^*.$$

So if we make a further continuation via the lower half of the complex plane, back to real values of s greater than $4m^2$, we obtain the complex conjugate of the physical amplitude:

$$A(s - i\epsilon, t) = [A(s + i\epsilon, t)]^*. \tag{1.32}$$

Therefore, for $s \geq 4m^2$ and $-s < t, u \leq 0$,

$$2i \, \mathrm{Im} \, A(s + i\epsilon, t) = A(s + i\epsilon, t) - A(s - i\epsilon, t) \tag{1.33}$$

where it is understood in this equation that we have to take the limit $\epsilon \to 0$. (By convention the imaginary part of the amplitude is defined to be real, as is evident from the factor $2i$.) The right hand side of (1.33) is called the s-channel discontinuity, denoted by $D_s(s,t,u)$.

Similar arguments can be applied to the physical t-channel and u-channel processes $1 + \bar{3} \to \bar{2} + 4$ and $1 + \bar{4} \to 3 + \bar{2}$. Thus there must be cuts along the

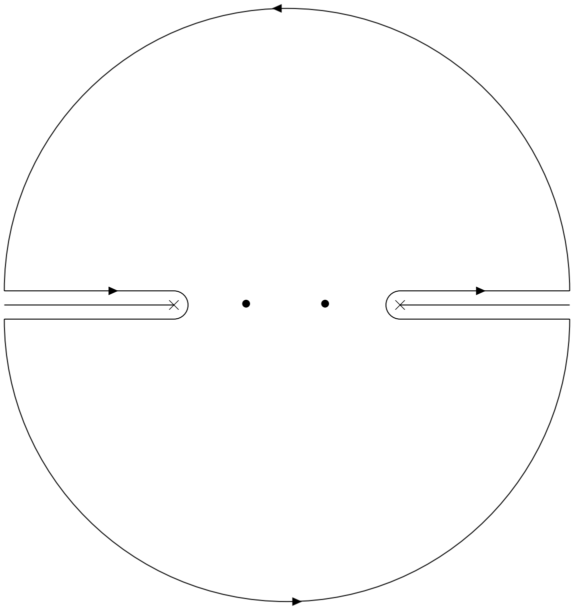

Figure 1.5. Contour of integration in the complex s'-plane

real positive t and u axes, with branch points at the appropriate physical thresholds in these channels, and possibly poles as well. Equivalently to (1.33) we define the t-channel and u-channel discontinuities by

$$\begin{aligned} D_t(s,t,u) &= A(s,t+i\epsilon) - A(s,t-i\epsilon) = 2i\,\mathrm{Im}\,A(s,t+i\epsilon) \\ &\qquad\qquad t > 4m^2 \text{ and } u,s \le 0 \\ D_u(s,t,u) &= A(s,u+i\epsilon) - A(s,u-i\epsilon) = 2i\,\mathrm{Im}\,A(s,u+i\epsilon) \\ &\qquad\qquad u > 4m^2 \text{ and } s,t \le 0 \end{aligned} \tag{1.34}$$

where again the limit $\epsilon \to 0$ is understood.

Knowing the analytic structure of an amplitude allows us to derive a "dispersion relation". We fix t and use the contour of integration shown in figure 1.5, which must be such that the point $s = s'$ is within it. Then $(s'-s)^{-1}A(s',t)$ is analytic within the contour except for a pole at $s' = s$, so that Cauchy's theorem tells us that the integral of this function is just the residue at the pole, which is $2\pi i A(s,t)$. Hence

$$A(s,t) = \frac{1}{2\pi i}\oint ds' \frac{A(s',t)}{s'-s} \tag{1.35}$$

with u (u') given in terms of s (s') and t by (1.3). Assume for the moment