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Spinning tops
A Course on Integrable Systems

Michèle Audin

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Acknowledgements

Among the works in which I have learned the most, there are three papers that I wish to mention here. I think that I was very lucky to start my initiation to integrable systems by reading two very beautiful papers that are not among the most cited, those of Verdier [84] and Griffiths [36]. On the one hand, Verdier had the brilliant idea of illustrating his exposition of the work of Adler and van Moerbeke by the example of the symmetric top. On the other hand, Griffiths’ main philosophical point was to look at a Lax equation, without specifying anything more. The advantage of this approach is that it allows one to start working without having first ingested loop algebras and the “AKS theorem”. Moreover, in this sober presentation, the role played by the eigenvectors of the Lax matrices is amply brought to light. The algebraic geometry related to these eigenvectors is extremely well described in Reyma’s paper [74] – which is the third paper to which I feel indebted.

The present text originates mainly from several talks I have given on the examples here, in particular on the work I have done jointly with R. Silhol [15] and from a graduate course I taught in Strasbourg in 1992-93, jointly with J.-Y. Méringol, on “Algebraic curves and integrable systems”. The first version, *Toiles, un cours sur les systèmes intégrables* was written at the end of 1993. I am very pleased to acknowledge the influence of the very clear survey of Reyma & Semenov-Tian-Shanski [77] on the present version.

I have learned a lot in discussions with Jean-Yves Méringol, Robert Silhol, and, especially, Alexei Reyman.

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Michele Audin
Strasbourg, January 30, 1996

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For the paperback edition, I have only corrected a few misprints.

Strasbourg, May 1, 1999