

Contents

Author’s Preface	<i>page</i> xvii
Historical Preface	xxi
Rise of the Academies	xxi
Abel and Galois	xxiv
Lie and Klein	xxvi
1870	xxviii
Lie’s Arrest	xxix
Gauss, Riemann, and the New Geometry	xxx
The Erlangen Program	xxxiv
Lie’s Career at Leipzig	xxxvi
A Falling Out	xxxvi
Lie’s Final Return to Norway	xxxvii
After 1900	xxxviii
The Ariadne Thread	xxxix
Suggested Reading	xl
1 Introduction to Symmetry	1
1.1 Symmetry in Nature	1
1.2 Some Background	3
1.3 The Discrete Symmetries of Objects	4
1.3.1 The Twelffold Discrete Symmetry Group of a Snowflake	4
1.4 The Principle of Covariance	10
1.5 Continuous Symmetries of Functions and Differential Equations	12
1.5.1 One-Parameter Lie Groups in the Plane	13

viii	<i>Contents</i>	
	1.5.2 Invariance of Functions, ODEs, and PDEs under Lie Groups	14
1.6	Some Notation Conventions	23
1.7	Concluding Remarks	25
1.8	Exercises	29
	References	31
2	Dimensional Analysis	33
2.1	Introduction	33
2.2	The Two-Body Problem in a Gravitational Field	34
2.3	The Drag on a Sphere	38
	2.3.1 Some Further Physical Considerations	43
2.4	The Drag on a Sphere in High-Speed Gas Flow	44
2.5	Buckingham's Pi Theorem – The Dimensional-Analysis Algorithm	47
2.6	Concluding Remarks	50
2.7	Exercises	50
	References	54
3	Systems of ODEs and First-Order PDEs; State-Space Analysis	55
3.1	Autonomous Systems of ODEs in the Plane	55
3.2	Characteristics	56
3.3	First-Order Ordinary Differential Equations	57
	3.3.1 Perfect Differentials	57
	3.3.2 The Integrating Factor; Pfaff's Theorem	59
	3.3.3 Nonsolvability of the Integrating Factor	60
	3.3.4 Examples of Integrating Factors	62
3.4	Thermodynamics; The Legendre Transformation	65
3.5	Incompressible Flow in Two Dimensions	68
3.6	Fluid Flow in Three Dimensions – The Dual Stream Function	69
	3.6.1 The Method of Lagrange	70
	3.6.2 The Integrating Factor in Three and Higher Dimensions	72
	3.6.3 Incompressible Flow in Three Dimensions	73
3.7	Nonlinear First-Order PDEs – The Method of Lagrange and Charpit	74
	3.7.1 The General and Singular Solutions	77
3.8	Characteristics in n Dimensions	79
	3.8.1 Nonlinear First-Order PDEs in n Dimensions	82
3.9	State-Space Analysis in Two and Three Dimensions	83
	3.9.1 Critical Points	84

<i>Contents</i>	<i>ix</i>
3.9.2 Matrix Invariants	84
3.9.3 Linear Flows in Two Dimensions	85
3.9.4 Linear Flows in Three Dimensions	88
3.10 Concluding Remarks	90
3.11 Exercises	91
References	95
4 Classical Dynamics	96
4.1 Introduction	96
4.2 Hamilton’s Principle	98
4.3 Hamilton’s Equations	101
4.3.1 Poisson Brackets	103
4.4 The Hamilton–Jacobi Equation	105
4.5 Examples	108
4.6 Concluding Remarks	119
4.7 Exercises	119
References	120
5 Introduction to One-Parameter Lie Groups	121
5.1 The Symmetry of Functions	121
5.2 An Example and a Counterexample	122
5.2.1 Translation along Horizontal Lines	122
5.2.2 A Reflection and a Translation	123
5.3 One-Parameter Lie Groups	123
5.4 Invariant Functions	125
5.5 Infinitesimal Form of a Lie Group	127
5.6 Lie Series, the Group Operator, and the Infinitesimal Invariance Condition for Functions	128
5.6.1 Group Operators and Vector Fields	130
5.7 Solving the Characteristic Equation $X\Psi[x] = 0$	130
5.7.1 Invariant Points	132
5.8 Reconstruction of a Group from Its Infinitesimals	132
5.9 Multiparameter Groups	134
5.9.1 The Commutator	135
5.10 Lie Algebras	136
5.10.1 The Commutator Table	137
5.10.2 Lie Subalgebras	138
5.10.3 Abelian Lie Algebras	138
5.10.4 Ideal Lie Subalgebras	139
5.11 Solvable Lie Algebras	139
5.12 Some Remarks on Lie Algebras and Vector Spaces	142

x	<i>Contents</i>	
5.13	Concluding Remarks	145
5.14	Exercises	145
	References	148
6	First-Order Ordinary Differential Equations	149
6.1	Invariant Families	149
6.2	Invariance Condition for a Family	152
6.3	First-Order ODEs; The Integrating Factor	155
6.4	Using Groups to Integrate First-Order ODEs	157
6.5	Canonical Coordinates	162
6.6	Invariant Solutions	166
6.7	Elliptic Curves	172
6.8	Criterion for a First-Order ODE to Admit a Given Group	174
6.9	Concluding Remarks	176
6.10	Exercises	176
	References	177
7	Differential Functions and Notation	178
7.1	Introduction	179
7.1.1	Superscript Notation for Dependent and Independent Variables	181
7.1.2	Subscript Notation for Derivatives	181
7.1.3	Curly-Brace Subscript Notation for Functions That Transform Derivatives	183
7.1.4	The Total Differentiation Operator	184
7.1.5	Definition of a Differential Function	185
7.1.6	Total Differentiation of Differential Functions	186
7.2	Contact Conditions	187
7.2.1	One Dependent and One Independent Variable	187
7.2.2	Several Dependent and Independent Variables	188
7.3	Concluding Remarks	189
7.4	Exercises	190
	References	190
8	Ordinary Differential Equations	191
8.1	Extension of Lie Groups in the Plane	191
8.1.1	Finite Transformation of First Derivatives	191
8.1.2	The Extended Transformation Is a Group	193
8.1.3	Finite Transformation of the Second Derivative	195
8.1.4	Finite Transformation of Higher Derivatives	195

	<i>Contents</i>	xi
8.1.5	Infinitesimal Transformation of the First Derivative	197
8.1.6	Infinitesimal Transformation of the Second Derivative	198
8.1.7	Infinitesimal Transformation of Higher-Order Derivatives	199
8.1.8	Invariance of the Contact Conditions	200
8.2	Expansion of an ODE in a Lie Series; The Invariance Condition for ODEs	201
8.2.1	What Does It Take to Transform a Derivative?	202
8.3	Group Analysis of Ordinary Differential Equations	203
8.4	Failure to Solve for the Infinitesimals That Leave a First-Order ODE Invariant	203
8.5	Construction of the General First-Order ODE That Admits a Given Group – The Ricatti Equation	204
8.6	Second-Order ODEs and the Determining Equations of the Group	207
8.6.1	Projective Group of the Simplest Second-Order ODE	208
8.6.2	Construction of the General Second-Order ODE that Admits a Given Group	213
8.7	Higher-Order ODEs	213
8.7.1	Construction of the General p th-Order ODE That Admits a Given Group	214
8.8	Reduction of Order by the Method of Canonical Coordinates	215
8.9	Reduction of Order by the Method of Differential Invariants	216
8.10	Successive Reduction of Order; Invariance under a Multiparameter Group with a Solvable Lie Algebra	217
8.10.1	Two-Parameter Group of the Blasius Equation	219
8.11	Group Interpretation of the Method of Variation of Parameters	228
8.11.1	Reduction to Quadrature	230
8.11.2	Solution of the Homogeneous Problem	231
8.12	Concluding Remarks	233
8.13	Exercises	233
	References	236
9	Partial Differential Equations	237
9.1	Finite Transformation of Partial Derivatives	238
9.1.1	Finite Transformation of the First Partial Derivative	238
9.1.2	Finite Transformation of Second and Higher Partial Derivatives	239
9.1.3	Variable Count	241
9.1.4	Infinitesimal Transformation of First Partial Derivatives	242

xii	<i>Contents</i>	
9.1.5	Infinitesimal Transformation of Second and Higher Partial Derivatives	243
9.1.6	Invariance of the Contact Conditions	245
9.2	Expansion of a PDE in a Lie Series – Invariance Condition for PDEs	247
9.2.1	Isolating the Determining Equations of the Group – The Lie Algorithm	247
9.2.2	The Classical Point Group of the Heat Equation	248
9.3	Invariant Solutions and the Characteristic Function	255
9.4	Impulsive Source Solutions of the Heat Equation	257
9.5	A Modified Problem of an Instantaneous Heat Source	263
9.6	Nonclassical Symmetries	269
9.6.1	A Non-classical Point Group of the Heat Equation	270
9.7	Concluding Remarks	272
9.8	Exercises	273
	References	275
10	Laminar Boundary Layers	277
10.1	Background	277
10.2	The Boundary-Layer Formulation	279
10.3	The Blasius Boundary Layer	281
10.3.1	Similarity Variables	282
10.3.2	Reduction of Order; The Phase Plane	284
10.3.3	Numerical Solution of the Blasius Equation as a Cauchy Initial-Value Problem	291
10.4	Temperature Gradient Shocks in Nonlinear Diffusion	293
10.4.1	First Try: Solution of a Cauchy Initial-Value Problem – Uniqueness	294
10.4.2	Second Try: Solution Using Group Theory	295
10.4.3	The Solution	298
10.4.4	Exact Thermal Analogy of the Blasius Boundary Layer	299
10.5	Boundary Layers with Pressure Gradient	301
10.6	The Falkner–Skan Boundary Layers	305
10.6.1	Falkner–Skan Sink Flow	310
10.7	Concluding Remarks	313
10.8	Exercises	313
	References	317
11	Incompressible Flow	318
11.1	Invariance Group of the Navier–Stokes Equations	318

<i>Contents</i>	<i>xiii</i>
11.2 Frames of Reference	321
11.3 Two-Dimensional Viscous Flow	323
11.4 Viscous Flow in a Diverging Channel	325
11.5 Transition in Unsteady Jets	329
11.5.1 The Impulse Integral	320
11.5.2 Starting-Vortex Formation in an Impulsively Started Jet	325
11.6 Elliptic Curves and Three-Dimensional Flow Patterns	353
11.6.1 Acceleration Field in the Round Jet	353
11.7 Classification of Falkner–Skan Boundary Layers	357
11.8 Concluding Remarks	360
11.9 Exercises	361
References	362
12 Compressible Flow	364
12.1 Invariance Group of the Compressible Euler Equations	365
12.2 Isentropic Flow	369
12.3 Sudden Expansion of a Gas Cloud into a Vacuum	370
12.3.1 The Gasdynamic–Shallow-Water Analogy	371
12.3.2 Solutions	372
12.4 Propagation of a Strong Spherical Blast Wave	375
12.4.1 Effect of the Ratio of Specific Heats	382
12.5 Compressible Flow Past a Thin Airfoil	384
12.5.1 Subsonic Flow, $M_\infty < 1$	386
12.5.2 Supersonic Similarity, $M_\infty > 1$	389
12.5.3 Transonic Similarity, $M_\infty \approx 1$	390
12.6 Concluding Remarks	391
12.7 Exercises	392
References	393
13 Similarity Rules for Turbulent Shear Flows	395
13.1 Introduction	395
13.2 Reynolds-Number Invariance	397
13.3 Group Interpretation of Reynolds-Number Invariance	401
13.3.1 One-Parameter Flows	401
13.3.2 Temporal Similarity Rules	403
13.3.3 Frames of Reference	404
13.3.4 Spatial Similarity Rules	405
13.3.5 Reynolds Number	407
13.4 Fine-Scale Motions	408
13.4.1 The Inertial Subrange	410

xiv	<i>Contents</i>	
13.5	Application: Experiment to Measure Small Scales in a Turbulent Vortex Ring	413
13.5.1	Similarity Rules for the Turbulent Vortex Ring	415
13.5.2	Particle Paths in the Turbulent Vortex Ring	417
13.5.3	Estimates of Microscales	419
13.5.4	Vortex-Ring Formation	420
13.5.5	Apparatus Design	422
13.6	The Geometry of Dissipating Fine-Scale Motion	424
13.6.1	Transport Equation for the Velocity Gradient Tensor	425
13.7	Concluding Remarks	437
13.8	Exercises	438
	References	444
14	Lie–Bäcklund Transformations	446
14.1	Lie–Bäcklund Transformations – Infinite Order Structure	447
14.1.1	Infinitesimal Lie–Bäcklund Transformation	449
14.1.2	Reconstruction of the Finite Lie–Bäcklund Transformation	452
14.2	Lie Contact Transformations	453
14.2.1	Contact Transformations and the Hamilton–Jacobi Equation	457
14.3	Equivalence Classes of Transformations	457
14.3.1	Every Lie Point Operator Has an Equivalent Lie–Bäcklund Operator	459
14.3.2	Equivalence of Lie–Bäcklund Transformations	459
14.3.3	Equivalence of Lie–Bäcklund and Lie Contact Operators	461
14.3.4	The Extended Infinitesimal Lie–Bäcklund Group	461
14.3.5	Proper Lie–Bäcklund Transformations	462
14.3.6	Lie Series Expansion of Differential Functions and the Invariance Condition	462
14.4	Applications of Lie–Bäcklund Transformations	464
14.4.1	Third-Order ODE Governing a Family of Parabolas	466
14.4.2	The Blasius Equation $y_{xxx} + yy_{xx} = 0$	471
14.4.3	A Particle Moving Under the Influence of a Spherically Symmetric Inverse-Square Body Force	474
14.5	Recursion Operators	478
14.5.1	Linear Equations	479
14.5.2	Nonlinear Equations	481

<i>Contents</i>	<i>xv</i>
14.6 Concluding Remarks	496
14.7 Exercises	496
References	497
15 Variational Symmetries and Conservation Laws	498
15.1 Introduction	498
15.1.1 Transformation of Integrals by Lie–Bäcklund Groups	499
15.1.2 Transformation of the Differential Volume	499
15.1.3 Invariance Condition for Integrals	500
15.2 Examples	504
15.3 Concluding Remarks	511
15.4 Exercises	512
References	513
16 Bäcklund Transformations and Nonlocal Groups	515
16.1 Two Classical Examples	517
16.1.1 The Liouville Equation	517
16.1.2 The Sine–Gordon Equation	519
16.2 Symmetries Derived from a Potential Equation; Nonlocal Symmetries	521
16.2.1 The General Solution of the Burgers Equation	522
16.2.2 Solitary-Wave Solutions of the Korteweg–de Vries Equation	533
16.3 Concluding Remarks	547
16.4 Exercises	548
References	550
Appendix 1 Review of Calculus and the Theory of Contact	552
A1.1 Differentials and the Chain Rule	552
A1.1.1 A Problem with Notation	553
A1.1.2 The Total Differentiation Operator	554
A1.1.3 The Inverse Total Differentiation Operator	555
A1.2 The Theory of Contact	555
A1.2.1 Finite-Order Contact between a Curve and a Surface	555
Appendix 2 Invariance of the Contact Conditions under Lie Point Transformation Groups	558
A2.1 Preservation of Contact Conditions – One Dependent and One Independent Variable	558
A2.1.1 Invariance of the First-Order Contact Condition	558

xvi	<i>Contents</i>	
	A2.1.2 Invariance of the Second-Order Contact Condition	559
	A2.1.3 Invariance of Higher-Order Contact Conditions	560
A2.2	Preservation of the Contact Conditions – Several Dependent and Independent Variables	562
	A2.2.1 Invariance of the First-Order Contact Condition	562
	A2.2.2 Invariance of the Second-Order Contact Conditions	564
	A2.2.3 Invariance of Higher-Order Contact Conditions	566
Appendix 3	Infinite-Order Structure of Lie–Bäcklund Transformations	569
A3.1	Lie Point Groups	569
A3.2	Lie–Bäcklund Groups	570
A3.3	Lie Contact Transformations	570
	A3.3.1 The Case $m > 1$	573
	A3.3.2 The Case $m = 1$	573
A3.4	Higher-Order Tangent Transformation Groups	574
A3.5	One Dependent Variable and One Independent Variable	580
A3.6	Infinite-Order Structure	581
	A3.6.1 Infinitesimal Transformation	582
	Reference	583
Appendix 4	Symmetry Analysis Software	584
A4.1	Summary of the Theory	586
A4.2	The Program	588
	A4.2.1 Getting Started	589
	A4.2.2 Using the Program	591
	A4.2.3 Solving the Determining Equations and Viewing the Results	593
A4.3	Timing, Memory and Saving Intermediate Data	594
	A4.3.1 Why Give the Output in the Form of Strings?	597
	A4.3.2 Summary of Program Functions	597
	References	600
	Author Index	601
	Subject Index	604