Generalized Riemann Problems in Computational Fluid Dynamics

MATANIA BEN-ARTZI and JOSEPH FALCOVITZ
The Hebrew University of Jerusalem

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2003

Printed in the United Kingdom at the University Press, Cambridge

Typeface Times Roman 10/13 pt. *System* LaTeX 2_ε [TB]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data

Ben-Artzi, Matania, 1948–

Generalized Riemann problems in computational fluid dynamics / Matania Ben-Artzi, Joseph Falcovitz.

p. cm. – (Cambridge monographs on applied and computational mathematics; 11)

Includes bibliographical references and index.

ISBN 0-521-77296-6

1. Fluid dynamics. 2. Riemann-Hilbert problems. I. Falcovitz, Joseph, 1937–. II. Title. III. Series.

QA911 .B36 2003

532′.05 – dc21 2002034811

ISBN 0 521 77296 6 hardback

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Introduction

This monograph deals with the *generalized Riemann problem* (GRP) of mathematical fluid dynamics and its application to computational fluid dynamics. It shows how the solution to this problem serves as a basic tool in the construction of a robust numerical scheme that can be successfully implemented in a wide variety of fluid dynamical topics. The flows covered by this exposition may be quite different in nature, yet they share some common features; they all belong to the class of compressible, inviscid, time-dependent flows. Fluid dynamical phenomena of this type often contain a number of smooth flow regions separated by singularities such as shock fronts, detonation waves, interfaces, and centered rarefaction waves. One must then address various computational issues related to this class of fluid dynamical problems, notably the "capturing" of discontinuities such as shock fronts, detonation waves, or interfaces; resolution of centered rarefaction waves where flow gradients are unbounded; and evaluation of flow variables in irregular computational cells at the intersection of a moving boundary surface with an underlying mesh.

From the mathematical point of view, the various systems of equations governing compressible, inviscid, time-dependent flow phenomena may all be characterized as systems of "(nonlinear) hyperbolic conservation laws."

Hyperbolic conservation laws (in one space variable) are systems of time-dependent partial differential equations. The most common problem associated with such systems is the *initial value problem* (henceforth IVP), which is the following: Given the values of the unknown functions at time $t = 0$ (as functions of the space variable $x \in \mathbb{R}$), use the equations to determine the evolution in time of those functions. When the unknown functions are defined over the whole real line $\mathbb{R}$, one often refers to the IVP as the "Cauchy Problem." In contrast, when the unknown functions are defined only over a finite interval $\mathcal{D} \subseteq \mathbb{R}$, suitable "boundary conditions" must be imposed at the endpoints of $\mathcal{D}$. From

the physical point of view the latter is clearly the more realistic case. Thus, for example, $\mathcal{D}$ can represent a pipe of finite length, in which one studies the evolution (in time) of flow variables subject to the system of fluid dynamical equations. In this case, the boundary conditions consist of influx and outflux requirements imposed on the pressure, velocity, etc. at the edges of the pipe.

The solutions to the problems considered here possess one common fundamental property, that of "finite propagation speed"; that is, the waves travel at finite speeds. Mathematically speaking, when a change in the initial data is confined to the neighborhood of some point A, it is "felt" by the solution at any other point B only after a certain amount of time, an amount that depends on the distance between the points. It is precisely this feature that allows the construction of "conservation law schemes" for the (numerical) approximation of the solutions.

Although this monograph focuses on the resolution of compressible, inviscid flow problems, and the construction of suitable conservation law schemes, an effort is made to place the treatment in the broader (theoretical and numerical) perspective of hyperbolic conservation laws. However, the necessary background material from physics is also included. We refer the reader to the classical book by R. Courant and K. O. Friedrichs [30] for a thorough discussion of the mathematical aspects of compressible flow. This book also discusses in detail the derivation of the flow equations from the underlying physical conservation laws. For mathematical treatments of hyperbolic conservation laws, we refer to the books by Courant and Hilbert [31], Evans [36], Hörmander [63], Lax [75], and Smoller [103].

To simplify the discussion, we consider primarily the associated Cauchy problem, thus avoiding the further mathematical complications introduced by boundary conditions. Naturally, when dealing with real flow examples, boundary conditions will be needed, and the ways in which they are introduced into the numerical scheme will be explained.

The origin of the subject matter of this monograph can be traced back some forty years, to the early days of computational fluid dynamics. It can best be described by the opening sentence to Section 12.15 of the book by Richtmyer and Morton [96]: "In 1959, Godunov described an ingenious method for one-dimensional problems with shocks."

Godunov's method, as much as it was recognized for its novelty and robustness, suffered from some significant drawbacks. It was Bram van Leer, some twenty years later, who, in an important breakthrough [112], has shown how to modify Godunov's original construction and, indeed, has made it possible to implement the method as the most efficient tool (to date) in this area of

computational fluid dynamics. In the simpler case of a scalar conservation law, these ideas will be explained in Chapter 3.

The monograph is divided into two parts, Part I (Basic Theory) and Part II (Numerical Implementation). Part I (Chapters 2–6) deals with the more basic aspects (theoretical and numerical) of systems of conservation laws and the development of the GRP method. Part II (Chapters 7–10) is devoted to several extensions (physical and geometric) of the GRP method for computational fluid dynamics. A more detailed discussion of the contents will follow. The reader will also find a brief summary at the beginning of each chapter.

In writing this monograph we have aimed at a wide readership, consisting not only of graduate students and researchers in applied mathematics but also of those working in various areas of physics and engineering. Yet, we have attempted to maintain a solid level of mathematical rigor. Notions such as "weak solutions" and "convergence of a scheme" are carefully introduced (Chapter 2) in suitable functional settings. We believe that, given the current mathematical level of modern numerical analysis, such concepts ought to be familiar to anyone working in this field. In particular, theorems related to the convergence of the Godunov scheme (in the scalar case) are proved in all mathematical detail (Section 2.2 and Appendix B). In this context we introduce (and compare numerically) some of the "classical" discrete schemes of hyperbolic equations, such as the Lax–Friedrichs and the Lax–Wendroff schemes. At the same time, our main objective in Chapters 2 and 3 is the introduction of the "high-resolution GRP scheme," by way of the Riemann and generalized Riemann problems. We refer to LeVeque [81] for introductory material on finite-difference schemes for conservation laws and to Richtmyer and Morton [96] for the general theory of finite-difference methods (primarily linear theory). A comprehensive survey of the convergence properties of finite-difference schemes to scalar conservation laws can be found in Godlewski and Raviart [54].

In Chapter 4 we introduce systems of conservation laws. The first section outlines the general mathematical background and can be skipped on first reading, as it is of a more mathematical nature. The physical systems of interest, those representing the basic conservation laws of compressible, inviscid flow in the "quasi-one-dimensional" setting, are introduced in the second section. This section is self-contained; the analysis of centered rarefaction waves, as well as the Rankine–Hugoniot shock conditions and the solution to the Riemann problem, is discussed in detail.

Chapter 5 is devoted to the analysis of the GRP in the context of the systems considered in Section 4.2. In Section 5.1 we study the solution to the linear GRP, which is the core of the GRP method. Given linear initial distributions

of the flow variables on the two sides of a jump discontinuity, we determine their instantaneous time derivatives (at that singularity). Van Leer's idea to use this solution for the refinement of Godunov's scheme is implemented in the development of the GRP scheme in Section 5.2.

Chapter 6 is devoted to an investigation of the GRP scheme for fluid dynamics. Numerical results are compared to analytical or asymptotic solutions for a variety of wave interaction problems.

In Chapter 7 we introduce, in rather general terms, the operator-splitting method of Strang. It enables us to extend the GRP algorithm to two-dimensional (2-D) settings, while retaining its second-order accuracy. Chapter 8 deals with further geometric extensions, such as (one-dimensional) "tracking" of singularities and (2-D) moving boundaries.

In Chapter 9 we consider a reacting flow system. The basic set of conservation laws is augmented by a chemical reaction-rate equation, thus providing a simple model of combustion. The GRP algorithm is applied to this extended system.

As a concluding (numerical) example for this monograph, we consider in Chapter 10 a case of wave interaction with a segment of decreasing cross-sectional area in a two-dimensional duct. The major (GRP) numerical approaches developed in Chapters 5, 7, and 8, namely the quasi-1-D approximation and the fully 2-D scheme, are applied to this case. A comparative study of the two solutions sheds light on the nature of the fluid dynamical interaction, as well as on the nature of the quasi-1-D approximation.

Finally, a comment about the numbering system in this book. For the reader's convenience, all theorems, remarks, definitions, claims, etc. within each chapter are sequentially numbered. Thus, for example, Remark 2.22 comes after Definition 2.21 and is followed by Example 2.23.