Introduction to Symmetry Analysis

Symmetry analysis based on Lie group theory is the most important method for solving nonlinear problems aside from numerical computation. The method can be used to find the symmetries of almost any system of differential equations, and the knowledge of these symmetries can be used to simplify the analysis of physical problems governed by the equations. This text offers a broad, self-contained introduction to the basic concepts of symmetry analysis and is intended primarily for first- and second-year graduate students in science, engineering, and applied mathematics. The book should also be of interest to researchers who wish to gain some familiarity with symmetry methods. The text emphasizes applications, and numerous worked examples are used to illustrate basic concepts.

Mathematica® based software for finding the Lie point symmetries and Lie–Bäcklund symmetries of differential equations is included on a CD, along with more than sixty sample notebooks illustrating applications ranging from simple, low-order ordinary differential equations to complex systems of partial differential equations. The notebooks are carefully coordinated with the text and are fully commented, providing the reader with clear, step-by-step instructions on how to work a wide variety of problems. The *Mathematica*® source code for the package is included on the CD.

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Author's Preface

This textbook grew out of the lectures for a course by the same name that I have given at Stanford University since the mid 1980s. The course is designed mainly for first- and second-year graduate students in science, engineering, and applied mathematics, although the material is presented in a form that should be understandable to an upper-level undergraduate with a background in differential equations. The students who come into the course usually have no knowledge of symmetry theory whatsoever and more often than not, they have been imbued with the notion that the only method available for solving nonlinear problems is numerical analysis. By the end of the course they recognize that symmetry analysis provides not an alternative to computation but a complementary analytical approach that is applicable to almost any system of differential equations they are likely to encounter.

The main goal is to teach the methods of symmetry analysis and to instill in the student a sense of confidence in dealing with complex problems. The central theme is that any time one is confronted with a physical problem and a set of equations to solve, the first step is to analyze the problem using dimensional analysis and the second is to use the methods of symmetry analysis to work out the Lie groups (symmetries) of the governing equations. This may or may not produce a simplification, but it will almost always bring clarity to the problem. Knowledge of symmetries provides the user with a certain point of view that enhances virtually any other solution method one may wish to employ. It is my firm belief that any graduate program in science or engineering needs to include a broad-based course on dimensional analysis *and* Lie groups. Symmetry analysis should be as familiar to the student as Fourier analysis, especially when so many unsolved problems are strongly nonlinear.

I have tried to design the book to serve this need and to help the reader become skilled at applying the techniques of symmetry analysis. Therefore, wherever possible I have included the detailed steps leading up to the main theoretical results. Most of the theory is developed in the first half of the book and a large number of relatively short worked examples are included to illustrate the concepts. I have also provided, in Chapters 9 through 16, a number of fully worked problems where the role of symmetry analysis as part of the complete solution of a problem is illustrated. Enough detail is included for the reader to follow each problem from formulation to solution. Although the worked problems are mostly taken from heat conduction, fluid mechanics and nonlinear wave propagation, they are designed to explore many of the different facets of symmetry analysis and therefore should be of general interest. Phase-space methods are established in Chapter 3 and used extensively throughout the rest of the text. The emphasis is on applications, and the exercises provided at the end of each chapter are designed to help the reader practice the material. They range in difficulty from straightforward applications of the theory to challenging research-level problems. Many of the exercises include a reference to the literature where details of the solution can be found.

Some of the exercises involving the identification of Lie symmetries should be worked by hand so that the reader has a chance to practice the Lie algorithm. When this is done, it will become quickly apparent that the calculational effort needed to find symmetries can be huge, even to reach a fairly simple result. To analyze by hand any but the simplest problem, a discouragingly large amount of effort is required. When the subject is approached this way, it is essentially inaccessible to all but the most dedicated workers. This is one of the main reasons why Lie theory was never adopted in the mainstream curricula in science and engineering. It is systematic and powerful but can be very cumbersome! Fortunately, we now live in an era when powerful symbol manipulation software packages are widely available. This allows the vast bulk of the routine effort in group analysis to be automated, bringing the whole subject completely within the reach of an interested student.

Several years ago, I developed a set of *Mathematica*®-based software tools to use in my course. The package is called **`IntroToSymmetry.m`** and has been exercised by about five generations of students working hundreds of problems of varying complexity. So far it seems to work quite well. The package is very good at constructing the list of determining equations of the group for pretty much any system of equations, and it contains limited tools for solving those equations. The main benefit the package brings to the book is a large number of worked examples and an opportunity for the reader to rapidly gain experience by working lots of problems on their own with the aid of the package. Details of the package are described in Appendix 4. The package with its source code is included on a CD along with more than sixty sample notebooks that are carefully coordinated with the examples and exercises in the text. The source code and the sample runs are fully and extensively commented. The sample runs range in complexity from single low-order ODEs, to systems of ODEs, to single PDEs, and large systems of PDEs. Applications to both closed and unclosed systems are illustrated along with the use of built-in *Mathematica* functions for manipulating the results.

There is a good deal more in this book than can be absorbed in one quarter. The course I teach at Stanford covers Chapters 1 to 3 and 5 to 10 with selected examples from Chapters 11 to 13 as well as some of the main results on nonlinear waves in Chapter 16. In a one-semester course I would include the lengthy but relatively self-contained Chapter 14 on Lie–Bäcklund symmetries (also called generalized symmetries). A full two-quarter sequence would include the material on Lagrangian dynamics in Chapters 4 and 15 as well as all of Chapter 16. In the second quarter, I would supplement the book with material from other sources on approximate symmetries and on discrete symmetries with applications to numerical analysis, two important topics that are not covered in the book. In addition, I would add more examples of variational symmetries that are covered briefly in Chapter 15.

My course is introductory in nature and this is the basis of the book title, but for the sake of completeness I have not shied away from including some material of an advanced nature. Appendicies 2 and 3 provide the background needed to understand the infinite order nature of Lie–Bäcklund groups. The development is straightforward but the math is fairly intricate and a little hard to follow. Yet this material underlies the whole treatment of such groups and without these appendicies, there would be a large hole in the development of the theory in Chapter 14. Chapter 9 ends with a rather advanced problem in nonlinear heat conduction and then a brief discussion of nonclassical symmetries, which is an active area of current research.

Chapters 10, 11 and 12 constitute a series of examples, all of which are drawn from heat conduction and fluid mechanics. The examples are intended to show in detail how groups relate to solutions, i.e., to show how symmetry analysis is really used. If the reader does not have a background in fluids, then I recommend three basic references: Van Dyke's collection of flow pictures called *An Album of Fluid Motion*, Batchelor's *An Introduction to Fluid Dynamics* and Liepmann and Roshko's *Elements of Gasdynamics* (complete references are given at the end of Chapters 10 and 12). These provide a good deal of the basic knowledge one may need to get through these examples. Chapter 13 is even more specialized and will probably appeal mainly to someone with a strong interest in turbulence. Yet it is hard to envision a text that does not touch on this important and complex subject where, in the absence of a complete theory, symmetry methods are an essential tool for solving problems.

I would like to acknowledge the fruitful association with my colleague Milton Van Dyke during the times we spent co-teaching his course on similitude when I first came to Stanford in 1978. That course was the predecessor of the one I teach today. I also want to express my appreciation to Nicholas Rott for our shared collaborations in fluid mechanics and for his words of encouragement on this project. Carl Wulfman from the University of the Pacific kindly reviewed an early version of the manuscript, and I thank him for his valuable and timely advice. Substantial parts of the book were developed while I was visiting the University of Notre Dame as the Melchor Chair Professor during the Fall semester of 1998, and I would like to thank Bob Nelson for supporting my visit. I would also like to thank my good friends at ND, Sam Paolucci, Mihir Sen, and Joe Powers, who sat through the course that semester and who provided so many helpful comments and suggestions. Thanks also to Nail Ibragimov for graciously hosting my visit to South Africa in December 1998. I would especially like to thank Stanford graduate students Alison Marsden and Jonathan Dirrenberger for their valuable comments and criticisms of the nearly final text. Finally, I would like to remember my good friend and a great scientist, Tony Perry (1937–2001) who first inspired my interest in the geometry of fluid flow patterns and whose insight was always on the mark. I wish we could meet just one more time in Melbourne to drink a few beers, shed a tear for Collingwood and share a laugh at the world.

Palo Alto, January 2001

This Book is dedicated to my loving family: Ruth, Alice, Kevin, and Tom.
(I promise to pick up all those piles of books, papers, photos, etc.,
and put them away now.)

Historical Preface

In his biography of Sophus Lie and Felix Klein, Isaak Yaglom (1988) states:

> It is my firm belief that of all of the general scientific ideas which arose in the 19th century and were inherited by our century, none contributed so much to the intellectual atmosphere of our time as the idea of symmetry.

Few would disagree with that statement. Lie and Klein were the main protagonists in the historical development of the theory of symmetry used widely today. It is a remarkable coincidence how these two late-19th-century mathematicians from such different backgrounds became friends and how their careers both diverged and remained intertwined throughout their lives. Toward the end of the 18th century, one of the main themes of European academic culture, fostered by the age of Enlightenment, was a remarkably free exchange of scholars and ideas across national boundaries. This freedom contributed mightily to the revolution in physics and mathematics that was to come in the 19th and early 20th centuries. It had such a profound effect on the development of the theory of symmetry that to understand the theory and its language one is compelled to understand its history. The story of Klein and Lie, as the reader will see, is virtually the story of the development of modern mathematics, and almost every mathematician whose name is familiar to our experience had an direct or indirect role in their remarkable careers.

Rise of the Academies

Mathematics and science in Europe lost its provincialism very quickly with the onset of the industrial revolution. By the early 17th century, mathematicians began to work in small groups, communicating their work through books or letters. Networks were created that coordinated and stimulated research. Marin

Mersenne in Paris collected and distributed new results to a number of correspondents, including Fermat, Descartes, Blaise Pascal, and Galileo, keeping them informed of the latest events. John Collins, the librarian of the Royal Society of London, founded in 1660, played this role among British mathematicians. The universities at the time provided relatively little support for research. Instead, state-supported academies, which tended to emphasize science and mathematics, usually with military applications in mind, carried out the most advanced research. At the urging of Jean Colbert, his chief minister, Louis XIV founded the French Academie Royale des Sciences in 1666. The Berlin Academy was founded in 1700, and the St. Petersburg Academy in 1724. After 1700, the movement to found learned societies spread throughout Europe and to the American colonies. At the same time, new journals were created, making possible prompt and, for the first time, wide dissemination of research results. The academy provided a forum for rigorous evaluation by peers, and it afforded scientists protection from political and religious persecution for their ideas. The separation of research from teaching distinguished the academy from the model of university-based science, which developed in the 19th century.

The preeminent mathematicians of the time, among them Leonhard Euler, Jean le Rond d'Alembert, and Joseph-Louis Lagrange, all followed careers in the academies in London, Paris, and St. Petersburg. The academies held meetings on a regular basis, published memoirs, organized scientific expeditions, and administered prize competitions on important scientific questions. One of the most famous of these is the subject of the beautifully written 1995 book *Longitude* by Dava Sobel. This was a £20,000 prize offered by Parliament in 1714 to anyone who could develop an accurate, practical method for determining longitude at sea. The board founded to oversee the prize included the president of the Royal Society as well as professors of mathematics from Oxford and Cambridge. This board was the predecessor of the modern government research and development agency. Over the one hundred years of its existence, the effort to measure longitude spun off other discoveries, including the determination of the mass of the Earth, the distance to the stars, and the speed of light.

During the period from 1700 to 1800, there was free movement of scholars across state boundaries, and the generally apolitical attitude they adopted toward their science contributed to an academic culture almost free of the national chauvinism that was rampant in state politics. Perhaps no one typifies this better than the great Italian–French mathematician Joseph-Louis Lagrange, born in 1736 at Turin, in what was then Sardinia-Piedmont. Lagrange was born into a well-to-do family of French origin on his father's side. His father was treasurer to the king of Sardinia. At 19, Lagrange was teaching mathematics at the artillery school of Turin, where he would later be one of the founders of the Turin academy. By 1761, he was recognized as one of the greatest living mathematicians and was awarded a prize by the Paris Academy of Sciences for a paper on the libration of the moon. In 1766, on the recommendation of the Swiss Leonhard Euler and the French Jean d'Alembert, he was invited by King Frederick II (the Great) of Prussia to become mathematical director of the Berlin Academy. During the next two decades, Lagrange wrote important papers on the three-body problem in celestial mechanics, differential equations, prime-number theory, probability, mechanics, and the stability of the solar system. In his 1770 paper, "Reflections on the Algebraic Resolution of Equations," he ushered in a new period in the theory of equations that would inspire Evariste Galois in his theory of groups four decades later. When Frederick died in 1787, Lagrange moved to Paris at the invitation of Louis XVI and took up residence in the Louvre, where, in 1788 on the eve of the French Revolution, he published his famous *Mécanique Analytique*. Napoleon honored him by making him a senator and a count of the empire. The quiet, unobtrusive mathematician whose career spanned the European continent was revered until his death in Paris in 1813, just as the post-Revolution Napoleonic era was approaching its end.

The French Revolution in 1789 was followed a decade later by the Napoleonic era and the final collapse, after the battle of Austerlitz in 1805, of the Holy Roman Empire, which in the words of Voltaire was "neither holy, nor Roman, nor an empire." This brought an end to the age of Enlightenment and the benevolent despotism of the royalist period, presaging a new era in scientific research and education. The new political order stimulated a rapid spread in scientific interest among all classes of society. Anyone with ability who wished to follow intellectual pursuits was encouraged to do so. There was a great increase in the number of students learning science and mathematics, and this drove an increased demand for teachers. These events forged a new relationship between teaching and research.

New centers of learning were established, and old ones revitalized. The French Revolution provoked a complete rethinking of education in France, and mathematics was given a prominent role. The Ecole Polytechnique was established in 1794 by Gaspard Monge (1746–1818) with Lagrange as its leading mathematician. It prepared students for the civil and military engineering schools of the Republic. Monge believed strongly that mathematics should serve the scientific and technical need of the state. To that end, he devised a syllabus that promoted descriptive geometry, which was useful in the design of forts, gun emplacements, and machines. The Ecole Polytechnique soon began to attract the best scientific minds in France. A similar center of learning was

On the left is a lithograph of Neils Henrik Abel, and on the right a sketch of Evariste Galois by his brother Alfred.

established by Carl Jacobi in Königsberg, Germany, in 1827. The University of Göttingen, founded in 1737 by George II of England in his role as the Elector of the Kingdom of Hanover, was beginning to attract students from all over Europe with a strong faculty in physics and mathematics. There was a fresh atmosphere of scientific excitement and curiosity. The creation of new knowledge flourished.

Abel and Galois

One of the central problems of mathematics research in the 19th century concerned the theory of equations. Ever since researchers in the 16th century had found rules giving the solutions of cubic and quartic equations in terms of the coefficients of the equations, formulas had unsuccessfully been sought for equations of the fifth and higher degrees. At the center of interest was the search for a formula that could express the roots of a quintic equation in terms of its coefficients using only the operations of addition, subtraction, multiplication, and division, together with the taking of radicals, as had been required for the solution of quadratic, cubic, and quartic equations. By 1770, Lagrange had analyzed all the methods for solving equations of degrees 2, 3, and 4, but he was not able to progress to higher order.

The first proof that the general quintic polynomial is not solvable by radicals was eventually offered in an 1824 paper by the Norwegian mathematician Niels

Henrik Abel (1802–1829). In Abel's time Norway was very provincial, and he was awarded a scholarship from the Norwegian government that enabled him to visit other mathematicians in Germany and France. His talent was recognized by the prominent German engineer and entrepreneur, August Leopold Crelle (1780–1855). Crelle had become very wealthy from the railroad business, and his belief in Abel and in his Swiss co-worker Jacob Steiner prompted Crelle to found the first specialized mathematical journal in Germany. The first volumes were filled with Abel's and Steiner's papers. Abel's work published in Crelle's journal attracted the attention of the famous Carl Jacobi, and because of the efforts of Jacobi and other German scientists, Abel was eventually appointed professor at Berlin University in 1828. Unfortunately, the official notice did not reach Kristiania (now Oslo) until several days after Abel's death from tuberculosis at the age of twenty-seven. Crelle's journal went on to play a major role in the development of German science.

Abel's proof was very limited in that it only asserted the absence of a general formula for the solution of every quintic equation in radicals. It did not indicate the special cases where the equation could be solved. This was taken up by Evariste Galois, who was a great admirer of Abel and who had studied Lagrange's work on the theory of equations and analytic functions while a student at the Lycée Louis-le-Grand in Paris. This well-known school included Robespierre and Victor Hugo among its graduates. Later, it would include the mathematician Charles Hermite (1822–1901), who in 1858 would publish the solution of the quintic equation in terms of elliptic functions.

Galois was born in the town of Bourg-la-Reine near Paris in 1811 and died in a duel over a broken affair with a woman in Paris in 1832. Although his childhood seems to have been quite happy, his late adolescence was marked by the suicide of his father and crushing disappointment at being twice rejected for entry by the Ecole Polytechnique. Like his father, he was a republican in an era when the monarchy was being restored in France. Galois spent much of the last few months of his life in and out of French prisons because of his fiery republican sentiments and for making death threats against the King. By 1830, the new bourgeois king, Louis-Philippe, had been forced to use repressive measures to counter numerous rebellions and attempts on his life. Rumors at the time suggested that Galois had been trapped into the duel. The exact circumstances of his death and whether there was any sort of conspiracy against him will probably never be known. In spite of his difficulties, Galois was able to create his theory of the solution of equations. The mixture of youth, mystery, tragedy, and his towering intellectual achievements make Galois one of the most romantic figures in the history of mathematics.

Galois created the first thorough classification of algebraic equations of the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0.$$

Galois, like Lagrange, Abel, and Gauss before him, asked: Is it possible to find the general solution of this equation by constructing resolvents (lower-order equations whose roots are rational functions of the roots of the original equation), i.e., can the equation be solved by means of radicals? *Galois theory*, as it is called today, provides a general criterion for the solvability of equations by resolvents, as well as a way to find the solutions. Galois also did extensive work on the integrals of algebraic functions of one variable (Abelian integrals). In addition, he left behind certain results that suggest he may have been a forerunner of Riemann. According to Klein in his *Development of Mathematics in the 19th Century*, Galois, in his farewell letter to his friend Chevalier, spoke of investigations into the "ambiguity of functions"; possibly foreshadowing the idea of Riemann surfaces and multiple connectivity.

Chevalier and Galois' younger brother, Alfred, copied Galois's mathematical papers and sent them to Carl Friedrich Gauss (1777–1855), who was by then the foremost mathematician in Europe, and to Jacobi, but received no response from either. The first to study them carefully was Joseph Liouville (1809–1882), who was professor at the Ecole Polytechnique. He became convinced of their importance and arranged to have them published in 1846, fourteen years after Galois's death. Today, virtually every mathematics department in the world offers a course in Galois theory. Galois introduced the concept of a *group* and defined many of the basic elements of group theory. Camille Jordan (1838–1922) recognized the many and varied applications of Galois's work and was inspired to write the first textbook on Galois theory, published in 1870. He introduced many of the main group-theoretic terms and ideas. In that same year, when he was preoccupied by his book and fascinated by group theory, two young postgraduate students from Berlin came to study with Jordan in Paris. They were Sophus Lie and Felix Klein.

Lie and Klein

Marius Sophus Lie was born in the vicarage at Eid in Nordfjord, Norway, on December 17, 1842. His father was rector of the parish at Eid, and his church still stands today, a few yards from the sea, next to a memorial to Norway's greatest mathematician. In 1851, the family moved to Moss on the Oslofjord, and he completed his secondary education in Kristiana (Oslo). During his youth, Lie

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Sophus Lie on the left and Felix Klein on the right. The photos depict the two men at mid-career.

was strongly encouraged to study for the ministry, and it was not until he was well into his twenties that he began to take a serious interest in mathematics. He published his first paper in Kristiana in 1869, when his talent was beginning to be recognized. On the basis of his paper, he was given a grant by the university to travel to Germany and France, where he was expected to study with eminent mathematicians of the time, develop his talent, and broaden his horizons. Based on the accounts of his friends, Yaglom describes Lie as

> . . . quite tall and physically very strong, with an open face and loud laugh; people who knew Lie often said he was their idea of a Viking, distinguished by rare candor and directness, always convivial with anyone who approached him, Lie produced an impression that did not correspond to his inner nature: actually he was very refined and easily hurt.

In 1869, he traveled to Berlin, which then was the center of the mathematical world, dominated by Kummer, Kronecker, and the head of the Berlin school of mathematics and a great proponent of strict mathematical rigor Karl Theodor Wilhelm Weierstrass (1815–1897). There he met the twenty-year-old Felix Klein, and their lifelong friendship began.

Christian Felix Klein was born in Dusseldorf in 1849 into the Prussian family of an official in the government finance department. Following his father's wishes, Klein studied at a classical gymnasium, where the emphasis was on ancient languages with very little attention to mathematics and science. Klein developed an intense dislike for the gymnasium, and this antipathy played an important role in his future views about teaching. After graduating from the gymnasium, Klein entered the university in Bonn, where he met Julius Plücker (1801–1868), who headed the departments of physics and mathematics. In 1866, at the age of seventeen, Klein became Plücker's assistant in the physics department. Two years later, Plücker died and to Klein fell the burden of publishing Plücker's unfinished works. Although this was an arduous and difficult task for someone so young, it contributed significantly to Klein's development as a mathematician. After Plücker's death, Klein lost his post as an assistant, left Bonn, and went to Göttingen, where he became acquainted with Rudolf Friedrich Alfred Clebsch (1833–1872), and then to Berlin, where he met the physicist Wilhelm Weber (1804–1891) and the mathematician Weierstrass.

Klein was known for a very physical way of thinking about mathematics, and his teaching was also characterized by a physical and graphical approach, and therefore a certain lack of rigor. Some of this can be traced back to Klein's rejection of the educational approach of the gymnasium, but much of it was the legacy left behind by Georg Friedrich Bernhard Riemann (1826–1866), whose ideas had profoundly influenced geometry. Riemann had succeeded Peter Gustav Lejuene Dirichlet (1805–1859) in 1859 as professor of mathematics at Göttingen, just as Dirichlet had succeeded the great Gauss four years earlier. Klein revered Riemann's work, but it did not appeal at all to the rigorous Weierstrass, and their relationship was not a warm one. Weierstrass had criticized Riemann and his friend Dirichlet and considered many of their results unproven or incorrect. Eventually, Klein would return to Göttingen to take Riemann's old position in 1886. In any case, Klein's arrival in Berlin in 1869 must have felt to Weierstrass a bit like the second coming of Riemann. Klein made up for the lack of intellectual contact with Weierstrass through his close collaboration with Lie. Although Klein was seven years younger than Lie, his experience publishing Plücker's work had matured him beyond his years. He was gregarious, with many powerful contacts, and he was a superb organizer. Thus, it was Klein who in later years would often provide the help Lie needed to advance his academic career.

1870

In February 1870, Lie traveled to Paris, and Klein arrived a few months later. There the students made contact with Camille Jordan and Gaston Darboux (1842–1917), who were then teaching at the Lycée Louis-le-Grande, where Galois had studied nearly half a century earlier. Under the influence of Jordan and Darboux, Klein and Lie continued their research, begun in Berlin, on the so-called *W*-curves, which are homogeneous curves that remain invariant under a certain group. *Homogeneous* curves are curves on which no point differs from any other. In plane Euclidean geometry, they are either straight lines or circles. The homogeneity of curves is related to the existence of a set of isometries that transform a curve into itself and each of its points into other points on the curve. In the case of a straight line, this self-isometry group is the group of translations along the direction of the line. For a circle, it is the group of rotations about the center of the circle. Another plane curve that has almost the same degree of homogeneity is the logarithmic spiral, whose equation in polar coordinates (r, φ) is $R = a^{\phi}$. The spiral allows self-similar transformations along itself. These transformations can be written in polar coordinates as $\tilde{R} = a^c R$ and $\tilde{\phi} = \phi + c$. They transform the point (R, ϕ) to the point $(\tilde{R}, \tilde{\phi})$, and the spiral onto itself. Lie and Klein posed the problem of finding each curve in the plane that has a group of projective transformations that map the curve into itself. They called such curves *W*-curves.

The search for *W*-curves was important for Lie's further research in that it led him to study one-parameter subgroups of the group of projective transformations, which would later play an important role in the construction of Lie algebras. In addition, he established the idea of an infinitesimal transformation. Their work together also laid the foundation for Klein's later research on the connection between projective geometry and its group of symmetries, which would eventually be the basis for his Erlangen program.

During his stay in Paris, Lie also discovered the concept of a contact transformation. *Contact transformations* are generalized surface mappings in a space that includes points and their tangents. The equations for tangency called contact conditions are preserved under the mapping. Lie's theory of contact transformations turns out to be intimately related to the identification of invariants of the motion in Hamiltonian mechanics.

Lie and Klein's collaboration in Berlin and Paris was motivated by their deep interest in the theory of groups and in the notion of symmetry. Afterward, their areas of scientific interest drifted apart.

Lie's Arrest

Their stay in Paris was abruptly ended by the outbreak of the Franco-Prussian War on July 18, 1870. Klein left Paris almost immediately for Germany, anticipating possible military service, and Lie left a month later. Lie was in no great hurry, and so he decided to walk to Milan and then hike home across Germany. While walking in a park in the town of Fontainebleau, his nordic looks attracted the attention of the police, and he was arrested as a German spy. The evidence against him included his letters from Klein and papers full of mathematical formulae. He spent a month in prison before Darboux arrived from Paris with his release order. Later, while traveling in Switzerland, he wrote of the experience:

> I have taken things truly philosophically. I think that a mathematician is well suited to be in prison. That doesn't mean that I accepted freedom philosophically. In truth, the sun has never seemed to shine so brightly, the trees have never seemed so green as they did yesterday when, as a free man, I walked to the railway station in Fontainebleau.

Lie had used his time in prison to work on his doctoral thesis, which was finally submitted to the University of Kristiana in June of 1871. In 1874, at the age of 32, Lie married Anna Sophie Birch. The younger Felix Klein had married a short time earlier. Both seem to have enjoyed happy marriages.

Gauss, Riemann, and the New Geometry

Klein became more interested in discrete groups, their relationship to geometry and the use of groups for the categorization of mathematical objects. Discrete groups of symmetries are also known as crystallographic groups, and the importance of such groups in the study of crystals was well recognized by the end of the 19th century. After returning from France and recovering from typhus, which kept him from military duty, Klein settled in Göttingen, close to Clebsch and Weber. It was there he made his most important scientific achievements.

To understand the context of these achievements, it is necessary to review another of the important threads in 19th-century mathematics. This was the intense interest in fundamental questions in classical geometry. Attention centered on the fifth postulate of Book I of the *Elements*, which Euclid had used to prove the existence of a unique parallel through a point to a given line. There are a number of equivalent ways of stating this postulate. Perhaps the simplest is as follows:

For each straight line L and point P outside of L there is only one straight line passing through P that does not intersect L.

This seemingly self-evident statement had troubled Greek, Islamic, and European geometers since antiquity. In contrast to the other of Euclid's postulates, the parallel axiom invokes a global concept of the geometry of space at infinity. Even Euclid avoided using the postulate and managed to prove his first twenty-eight propositions without it. The Italian Jesuit Girolamo Saccheri (1667–1733) attempted to prove the parallel axiom by showing that all possible alternatives produce absurd results. By the end of his efforts, he had, in effect, discovered several of the theorems of a new geometry. However, he refused to accept this and remained devoutly convinced that Euclid's geometry was the only true way to describe space.