PATTERN FORMATION AND DYNAMICS IN NONEQUILIBRIUM SYSTEMS

Many exciting frontiers of science and engineering require understanding of the spatiotemporal properties of sustained nonequilibrium systems such as fluids, plasmas, reacting and diffusing chemicals, crystals solidifying from a melt, heart muscle, and networks of excitable neurons in brains.

This introductory textbook for graduate students in biology, chemistry, engineering, mathematics, and physics provides a systematic account of the basic science common to these diverse areas. This book provides a careful pedagogical motivation of key concepts, discusses why diverse nonequilibrium systems often show similar patterns and dynamics, and gives a balanced discussion of the role of experiments, simulation, and analytics. It contains numerous illustrative worked examples, and over 150 exercises.

This book will also interest scientists who want to learn about the experiments, simulations, and theory that explain how complex patterns form in sustained nonequilibrium systems.

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"This book by Cross and Greenside presents a comprehensive introduction to an important area of natural science, and assembles in one volume the essential conceptual, theoretical, and experimental tools a serious student will need to obtain a modern understanding of pattern formation outside of equilibrium. The masterful 50-page Introduction lays out the essential questions and provides motivation to the reader to explore the subsequent chapters, beginning with simple ideas and growing progressively in mathematical sophistication and physical depth. Careful attention is paid to the relationship between the theoretical methods and controlled laboratory experiments or numerical simulations. I can highly recommend this book to any student or researcher interested in a deepened understanding of nonequilibrium spatiotemporal patterns."

Pierre Hohenberg, New York University

"This book gives an excellent didactic introduction to pattern formation in spatially extended systems. It can serve both as the basis for an advanced undergraduate or graduate course as well as a reference. It is one of those books that will never outlive its usefulness. It is a must for anyone interested in nonlinear, nonequilibrium physics."

Eberhard Bodenschatz, MPI for Dynamics and Self-Organization,
University of Goettingen, Cornell University

"This book fills a long-standing need, and is certain to be an instant classic. The physics of pattern forming systems is diverse but the theoretical core of the subject, along with many of the most important applications, can be learned from this splendid book. It is bound to be a key text for courses, as well as a much cited reference."

Stephen Morris, University of Toronto

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To our families

Katy, Colin and Lynn
Peyton, Arthur and Noel

Contents

Preface *page* xiii
1 Introduction **1**
1.1 The big picture: why is the Universe not boring? 2
1.2 Convection: a first example of a nonequilibrium system 3
1.3 Examples of nonequilibrium patterns and dynamics 10
1.3.1 Natural patterns 10
1.3.2 Prepared patterns 20
1.3.3 What are the interesting questions? 35
1.4 New features of pattern-forming systems 38
1.4.1 Conceptual differences 38
1.4.2 New properties 43
1.5 A strategy for studying pattern-forming nonequilibrium systems 44
1.6 Nonequilibrium systems not discussed in this book 48
1.7 Conclusion 49
1.8 Further reading 50
2 Linear instability: basics **56**
2.1 Conceptual framework for a linear stability analysis 57
2.2 Linear stability analysis of a pattern-forming system 63
2.2.1 One-dimensional Swift–Hohenberg equation 63
2.2.2 Linear stability analysis 64
2.2.3 Growth rates and instability diagram 67
2.3 Key steps of a linear stability analysis 69
2.4 Experimental investigations of linear stability 70
2.4.1 General remarks 70
2.4.2 Taylor–Couette instability 74

2.5 Classification for linear instabilities of a uniform state 75
2.5.1 Type-I instability 77
2.5.2 Type-II instability 79
2.5.3 Type-III instability 80
2.6 Role of symmetry in a linear stability analysis 81
2.6.1 Rotationally invariant systems 82
2.6.2 Uniaxial systems 84
2.6.3 Anisotropic systems 86
2.6.4 Formal discussion 86
2.7 Conclusions 88
2.8 Further reading 88
3 Linear instability: applications 96
3.1 Turing instability 96
3.1.1 Reaction–diffusion equations 97
3.1.2 Linear stability analysis 99
3.1.3 Oscillatory instability 108
3.2 Realistic chemical systems 109
3.2.1 Experimental apparatus 109
3.2.2 Evolution equations 110
3.2.3 Experimental results 116
3.3 Conclusions 119
3.4 Further reading 120
4 Nonlinear states 126
4.1 Nonlinear saturation 129
4.1.1 Complex amplitude 130
4.1.2 Bifurcation theory 134
4.1.3 Nonlinear stripe state of the Swift–Hohenberg equation 137
4.2 Stability balloons 139
4.2.1 General discussion 139
4.2.2 Busse balloon for Rayleigh–Bénard convection 147
4.3 Two-dimensional lattice states 152
4.4 Non-ideal states 158
4.4.1 Realistic patterns 158
4.4.2 Topological defects 160
4.4.3 Dynamics of defects 164
4.5 Conclusions 165
4.6 Further reading 166

5 Models **173**
5.1 Swift–Hohenberg model 175
5.1.1 Heuristic derivation 176
5.1.2 Properties 179
5.1.3 Numerical simulations 183
5.1.4 Comparison with experimental systems 185
5.2 Generalized Swift–Hohenberg models 187
5.2.1 Non-symmetric model 187
5.2.2 Nonpotential models 188
5.2.3 Models with mean flow 188
5.2.4 Model for rotating convection 190
5.2.5 Model for quasicrystalline patterns 192
5.3 Order-parameter equations 192
5.4 Complex Ginzburg–Landau equation 196
5.5 Kuramoto–Sivashinsky equation 197
5.6 Reaction–diffusion models 199
5.7 Models that are discrete in space, time, or value 201
5.8 Conclusions 201
5.9 Further reading 202
6 One-dimensional amplitude equation **208**
6.1 Origin and meaning of the amplitude 211
6.2 Derivation of the amplitude equation 214
6.2.1 Phenomenological derivation 214
6.2.2 Deduction of the amplitude-equation parameters 217
6.2.3 Method of multiple scales 218
6.2.4 Boundary conditions for the amplitude equation 219
6.3 Properties of the amplitude equation 221
6.3.1 Universality and scales 221
6.3.2 Potential dynamics 224
6.4 Applications of the amplitude equation 226
6.4.1 Lateral boundaries 226
6.4.2 Eckhaus instability 230
6.4.3 Phase dynamics 234
6.5 Limitations of the amplitude-equation formalism 237
6.6 Conclusions 238
6.7 Further reading 239
7 Amplitude equations for two-dimensional patterns **244**
7.1 Stripes in rotationally invariant systems 246
7.1.1 Amplitude equation 246
7.1.2 Boundary conditions 248

7.1.3 Potential 249
7.1.4 Stability balloon 250
7.1.5 Phase dynamics 252
7.2 Stripes in anisotropic systems 253
7.2.1 Amplitude equation 253
7.2.2 Stability balloon 254
7.2.3 Phase dynamics 255
7.3 Superimposed stripes 255
7.3.1 Amplitude equations 256
7.3.2 Competition between stripes and lattices 261
7.3.3 Hexagons in the absence of field-inversion symmetry 264
7.3.4 Spatial variations 269
7.3.5 Cross-stripe instability 270
7.4 Conclusions 272
7.5 Further reading 273
8 Defects and fronts 279
8.1 Dislocations 281
8.1.1 Stationary dislocation 283
8.1.2 Dislocation dynamics 285
8.1.3 Interaction of dislocations 289
8.2 Grain boundaries 290
8.3 Fronts 296
8.3.1 Existence of front solutions 296
8.3.2 Front selection 303
8.3.3 Wave-number selection 307
8.4 Conclusions 309
8.5 Further reading 309
9 Patterns far from threshold 315
9.1 Stripe and lattice states 317
9.1.1 Goldstone modes and phase dynamics 318
9.1.2 Phase diffusion equation 320
9.1.3 Beyond the phase equation 327
9.1.4 Wave-number selection 331
9.2 Novel patterns 337
9.2.1 Pinning and disorder 338
9.2.2 Localized structures 340
9.2.3 Patterns based on front properties 342
9.2.4 Spatiotemporal chaos 345
9.3 Conclusions 352
9.4 Further reading 353

10 Oscillatory patterns 358
10.1 Convective and absolute instability 360
10.2 States arising from a type-III-o instability 363
10.2.1 Phenomenology 363
10.2.2 Amplitude equation 365
10.2.3 Phase equation 368
10.2.4 Stability balloon 370
10.2.5 Defects: sources, sinks, shocks, and spirals 372
10.3 Unidirectional waves in a type-I-o system 379
10.3.1 Amplitude equation 380
10.3.2 Criterion for absolute instability 382
10.3.3 Absorbing boundaries 383
10.3.4 Noise-sustained structures 384
10.3.5 Local modes 386
10.4 Bidirectional waves in a type-I-o system 388
10.4.1 Traveling and standing waves 389
10.4.2 Onset in finite geometries 390
10.4.3 Nonlinear waves with reflecting boundaries 392
10.5 Waves in a two-dimensional type-I-o system 393
10.6 Conclusions 395
10.7 Further reading 396
11 Excitable media 401
11.1 Nerve fibers and heart muscle 404
11.1.1 Hodgkin–Huxley model of action potentials 404
11.1.2 Models of electrical signaling in the heart 411
11.1.3 FitzHugh–Nagumo model 413
11.2 Oscillatory or excitable 416
11.2.1 Relaxation oscillations 419
11.2.2 Excitable dynamics 420
11.3 Front propagation 421
11.4 Pulses 424
11.5 Waves 426
11.6 Spirals 430
11.6.1 Structure 430
11.6.2 Formation 436
11.6.3 Instabilities 437
11.6.4 Three dimensions 439
11.6.5 Application to heart arrhythmias 439
11.7 Further reading 441

12 Numerical methods **445**
12.1 Introduction 445
12.2 Discretization of fields and equations 447
12.2.1 Finitely many operations on a finite amount of data 447
12.2.2 The discretization of continuous fields 449
12.2.3 The discretization of equations 451
12.3 Time integration methods for pattern-forming systems 457
12.3.1 Overview 457
12.3.2 Explicit methods 460
12.3.3 Implicit methods 465
12.3.4 Operator splitting 470
12.3.5 How to choose the spatial and temporal resolutions 473
12.4 Stationary states of a pattern-forming system 475
12.4.1 Iterative methods 476
12.4.2 Newton's method 477
12.5 Conclusion 482
12.6 Further reading 485
Appendix 1 Elementary bifurcation theory 496
Appendix 2 Multiple-scales perturbation theory 503
Glossary 520
References 526
Index 531

Preface

This book is an introduction to the patterns and dynamics of sustained nonequilibrium systems at a level appropriate for graduate students in biology, chemistry, engineering, mathematics, physics, and other fields. Our intent is for the book to serve as a second course that continues from a first introductory course in nonlinear dynamics. While a first exposure to nonlinear dynamics traditionally emphasizes how systems evolve in time, this book addresses new questions about the spatiotemporal structure of nonequilibrium systems. Students and researchers who succeed in understanding most of the material presented here will have a good understanding of many recent achievements and will be prepared to carry out original research on related topics.

We can suggest three reasons why nonequilibrium systems are worthy of study. First, observation tells us that most of the Universe consists of nonequilibrium systems and that these systems possess an extraordinarily rich and visually fascinating variety of spatiotemporal structure. So one answer is sheer basic curiosity: where does this rich structure come from and can we understand it? Experiments and simulations further tell us that many of these systems – whether they be fluids, granular media, reacting chemicals, lasers, plasmas, or biological tissues – often have *similar* dynamical properties. This then is the central scientific puzzle and challenge: to identify and to explain the similarities of different nonequilibrium systems, to discover unifying themes, and, if possible, to develop a quantitative understanding of experiments and simulations.

A second reason for studying nonequilibrium phenomena is their importance to technology. Although the many observed spatiotemporal patterns are often interesting in their own right, an understanding of such patterns – e.g. being able to predict when a pattern will go unstable or knowing how to select a pattern that maximizes some property like heat transport – is often important technologically. Representative examples are growing pure crystals, designing a high-power coherent laser, improving yield and selectivity in chemical synthesis, and inventing new electrical control techniques to prevent epilepsy or a heart attack. In these and other cases such as forecasting the weather or predicting earthquakes, improvements in the design, control, and prediction of nonequilibrium systems are often limited by our incomplete understanding of sustained nonequilibrium dynamics.

Finally, a third reason for learning the material in this book is to develop specific conceptual, mathematical, and numerical skills for understanding complex phenomena. Many nonequilibrium systems involve continuous media whose quantitative description is given in terms of nonlinear partial differential equations. The solutions of such equations can be difficult to understand (e.g. because they may evolve nonperiodically in time and be simultaneously disordered in space), and questions such as "Is the output from this computer simulation correct?," "Is this simulation producing the same results as my experimental data?" or "Is experimental noise relevant here?" may not be easily answered. As an example, one broadly useful mathematical technique that we discuss and use several times throughout the book is multiscale perturbation theory, which leads to so-called "amplitude equations" that provide a quantitatively useful reduction of complex dynamics. We also discuss the role of numerical simulation, which has some advantages and disadvantages compared to analytical theory and experimental investigation.

To help the reader master the various conceptual, mathematical, and numerical skills, the book has numerous worked examples that we call etudes. By analogy to a musical etude, which is a composition that helps a music student master a particular technique while also learning a piece of artistic value, our etudes are one- to two-page long worked examples that illustrate a particular idea and that also try to provide a non-trivial application of the idea.

Although this book is intended for an interdisciplinary audience, it is really a physics book in the following sense. Many of the nonequilibrium systems in the Universe, for example a germ or a star, are simply too complex to analyze directly and so are ill-suited for discovering fundamental properties upon which a general quantitative understanding can be developed. In much of this book, we follow a physics tradition of trying to identify and study simple idealized experimental systems that also have some of the interesting properties observed in more complex systems.

Thus instead of studying the exceedingly complex dynamics of the Earth's weather, which would require in turn understanding the effects of clouds, the solar wind, the coupling to oceans and ice caps, the topography of mountains and forest, and the effects of human industry, we instead focus our experimental and theoretical attention on enormously simplified laboratory systems. One example is Rayleigh–Bénard convection, which is a fluid experiment consisting of a thin horizontal layer of a pure fluid that is driven out of equilibrium by a vertical temperature difference that is constant in time and uniform in space. Another is a mixture of reacting and diffusing chemicals in a thin layer of gel, with reservoirs of chemicals to sustain the reaction. The bet is then that to understand aspects of what is going on in the weather or in an epileptic brain, it will be useful to explore some basic questions first for convection and other well-controlled laboratory systems. Similarly, as we discuss later in the book, there are conceptual, mathematical, and computational advantages if one studies simplified and reduced mathematical models such as the Swift–Hohenberg and complex Ginzburg–Landau equations when trying to understand the much more difficult partial differential equations that describe physical systems quantitatively. The experiments, models, simulations, and theory discussed in this book – especially the numerous comparisons of theory and simulation with experiment – will give the reader valuable insights and confidence about how to think about the more complex systems that are closer to their interests.

As background, readers of this book should know the equivalent of an introductory nonlinear dynamics course at the level of Strogatz's book [99]. Readers should feel comfortable with concepts such as phase space, dissipation, attractors (fixed points, limit cycles, tori, and strange attractors), basins of attractors, the basic bifurcations (super- and subcritical, saddle-node, pitchfork, transcritical, Hopf), linear stability analysis of fixed points, Lyapunov exponents, and fractal dimensions. A previous exposure to thermodynamics and to fluid dynamics at an undergraduate level will be helpful but is not essential and can be reviewed as needed. The reader will need to be competent in using multivariate calculus, linear algebra, and Fourier analysis at a junior undergraduate level. Several appendices in this book provide concise reviews of some of this prerequisite material, but only on those parts that are important for understanding the text.

There is too much material in this book for a single semester class so we give here some suggestions of what material could be covered, based on several scenarios of how the book might be used.

The first six chapters present the basic core material and should be covered in most classes for which this book is a main text. By the end of Chapter 6, most of the main ideas have been introduced, at least qualitatively. The successive chapters present more advanced material that can be discussed selectively. For example, those particularly interested in the systematic treatment of stationary patterns may choose to complete the semester by studying all or parts of Chapters 7 and 8, which provide quantitative discussions of two-dimensional patterns and localized structures, and Chapter 9 which is a more qualitative discussion of stationary patterns far from onset. For a less mathematical approach, it is possible to leave out the more technical Chapters 7 and 8 and move straight to Chapter 9 although we recommend including the first three subsections of Section 7.3 on the central question of the competition between stripes, two-dimensional lattices, and quasiperiodic patterns (these sections can be read independently of the remainder of the chapter). If the interest is more in dynamical phenomena, such as oscillations, propagating pulses, and waves (which may be the case if applying the ideas to signalling phenomena in biology is a goal), the class may choose to skip Chapters 7–9, pausing briefly to study Section 8.3 on fronts, and move immediately to Chapters 10 and 11 on oscillatory patterns and excitable media. Numerical simulations are vital to many aspects of the study of pattern formation, and to nonlinear dynamics in general, and so any of the above suggestions may include all or parts of Chapter 12.

In learning about nonequilibrium physics and in writing this book, the authors have benefited from discussions with many colleagues and students. We would like to thank Philip Bayly, Bob Behringer, Eshel Ben-Jacob, Eberhard Bodenschatz, Helmut Brand, Hugues Chaté, Peilong Chen, Elizabeth Cherry, Keng-Hwee Chiam, Bill Coughran Jr., Peter Daniels, David Egolf, Bogdan Epureanu, Paul Fischer, Jerry Gollub, Roman Grigoriev, James Gunton, Craig Henriquez, Alain Karma, Kihong Kim, Paul Kolodner, Lorenz Kramer, Andrew Krystal, Eugenia Kuo, Ming-Chih Lai, Herbert Levine, Ron Lifshitz, Manfred Lücke, Paul Manneville, Dan Meiron, Steve Morris, Alan Newell, Corey O'Hern, Mark Paul, Werner Pesch, Joel Reisman, Hermann Riecke, Sam Safran, Janet Scheel, Berk Sensoy, Boris Shraiman, Eric Siggia, Matt Strain, Cliff Surko, Harry Swinney, Shigeyuki Tajima, Gerry Tesauro, Yuhai Tu, Wim van Saarloos, and Scott Zoldi. We would like to express our appreciation to John Bechhoefer, Roman Grigoriev, Pierre Hohenberg, Steven Morris, and Wim Van Saarloos for helpful comments on early drafts of this book. And we would like especially to thank Guenter Ahlers and Pierre Hohenberg for many enjoyable and inspiring discussions over the years.

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Readers can find supplementary material on the book website at

`http://mcc.caltech.edu/pattern-formation-book/`,

or

`http://www.phy.duke.edu/~hsg/pattern-formation-book/`.

A list of errata will also be available on these websites.

We would be grateful to readers if they could forward to us comments they might have about the material and its presentation. These comments can be e-mailed to us at `mcc@caltech.edu` or `hsg@phy.duke.edu`.