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Representations of Groups

A Computational Approach

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Preface

The representation theory of finite groups was developed around 1900 by Frobenius, Schur and Burnside. The theory was first concerned with representing groups by groups of matrices over the complex numbers or a field of characteristic zero. Representations over fields of prime characteristic, called "modular representations" (as opposed to "ordinary" ones), were considered somewhat later, and its theory began with fundamental papers by R. Brauer starting in 1935. Despite its age, the representation theory of finite groups is still developing vigorously and remains a very attractive area of research. In fact, the theory is notorious for its large number of longstanding open problems and challenging conjectures. The availability of computers, the development of algorithms and computer algebra systems within the last few decades have had some impact on representation theory, perhaps most noticeable by the appearance of the *ATLAS of Finite Groups* in 1985 (see [38]). Note that we refer to this as the ATLAS in the text.

The present book gives an introduction into representation theory of finite groups with some emphasis on the computational aspects of the subject. The book grew out of some sets of courses that the senior of the authors has given at Aachen University since the early 1990s. It was our experience that many students appreciated having many concrete examples illustrating the abstract theory.

The range of examples in the area is rather limited if one restricts oneself to paper and pencil work, but can be greatly enhanced by using a computer algebra system such as GAP or MAGMA. For the examples and exercises in this book we have chosen GAP, which can be freely obtained from `http://www.gap-system.org` and for which the source code is publicly available. We did not want to use these systems as mysterious black boxes, so we have explained along with the theory the most important algorithms in the field, leaving out technical details or complexity questions altogether. Instead we have included in some examples (commented and sometimes edited) GAP-code mainly to give the unexperienced reader an impression of how easily most of these calculations can be done. The complete (and unedited) GAP-code and all special GAP-programs used for the examples and exercises in this book appear on the homepage of this book: `http://www.math.rwth-aachen.de/~RepresentationsOfGroups`. Here one can also find solutions to some of the exercises in the book. It is also planned to include additional material and a list of errata.

We have treated ordinary and modular representation theory together, not only because this seems to be economic, but also since there are so many interactions. For that reason we also did not refrain from occasional forward references, in particular in the examples given.

The book presupposes some knowledge on basic topics in abstract algebra, such as the Sylow theorems and occasionally some Galois theory. Although modules over algebras are defined, some familiarity with these notions will be assumed. Also the reader should be familiar with linear algebra, including normal forms of matrices. Tensor products are introduced, but for most of their basic properties we refer to standard text books in algebra.

The first chapter introduces the basic notions of representation theory and describes as examples the representations of cyclic groups and algebras. Permutation modules are then discussed in some detail, because of their importance for practical examples. Simple modules are treated in Section 1.3, including Norton's criterion and algorithms for proving or disproving simplicity often referred to under the key-word "Meataxe." The chapter also includes the relevant material on projective modules and blocks.

Ordinary characters of finite groups are treated in the second chapter. We give several applications of characters in different areas of algebra. We also include several algorithms for computing character tables of groups, such as the Dixon–Schneider algorithm, which can be applied if one can compute within a group sufficiently well to find the conjugacy classes. Other methods apply when one knows just the centralizer orders and perhaps a few characters. The chapter finishes with an example in which the character table of a simple group is computed using only the order of the group.

The third chapter covers the interplay between representations of groups and subgroups, which is, of course, vital for the representation theory of groups. We include a section on tables of marks as introduced already by Burnside. Marks can be interpreted as extensions of permutation characters, and tables of marks may be extremely useful when dealing with particular groups. Of course, Clifford theory and projective representations are covered. We also describe B. Fischer's method of Clifford matrices to compute character tables of certain group extensions which often occur as local (or maximal) subgroups of simple groups. The method is somewhat technical and perhaps best explained by giving examples, which we do. The chapter closes with Brauer's characterization of characters including some applications.

The last and longest chapter is devoted to modular representation theory. Our use of p-modular systems differs slightly from the one in the literature and we introduce standard p-modular systems in order to arrive at uniquely defined Brauer character tables, which we introduce in Section 4.2 using Conway polynomials. This is important, especially when one is dealing with Brauer characters of different groups at the same time, which very often is the case in concrete problems. We give examples for computing Brauer character tables using basic sets and other methods, in particular condensation. The chapter includes an exposition on Brauer's main theorems on blocks and the Green correspondence. Here the book is not entirely self-contained. There are a few cases where we omit proofs and give instead proper references to the literature, for instance Green's indecomposability theorem in Section 4.8. Trivial source modules are treated including Conlon's Induction Theorem, which was already used in an example in Section 3.5. They also provide easy examples for the Green correspondence. We don't give a proof of Brauer's theorem on blocks of defect one, but instead include some applications. Modular representations of p-solvable groups are treated only to an extent to be able to prove the Fong–Swan theorem and to explain the connection between the $k(GV)$-problem and Brauer's $\mathrm{k}(B)$-problem for solvable groups. Modular representation theory abounds in longstanding open problems and conjectures. The final section mentions some of the most famous ones and verifies them in some examples.

Finally we would like to point to the literature we have used and also alternative treatments which might be useful for the reader. The most comprehensive monographs on representation theory of groups are found in [41] and [42]. Students who might find our first section a bit daunting should perhaps consult some slower-paced introductory text such as [3], [73] or [97]. A standard text mainly on ordinary character theory is [92]. Concerning modular representation theory, [1] is an accessible introduction dealing only with modules, where [68] deals only with characters. All aspects of modular representation theory are covered in [57], which also contains a full proof of the Brauer–Dade theorem on blocks with cyclic defect groups. References [125] and [126], which we have used frequently, are not as comprehensive but are more easily accessible. For alternative treatments see [50], [51], [109] and [10]. Of course there are many topics that we have barely touched, or omitted altogether. We mention just two, the theory of exceptional characters, for which we refer to [33], and the representation theory of finite groups of Lie type covered in [25].

This book would not have been written without the existence and availability of the GAP system. So we wish to thank the whole GAP team for its work and in particular our colleague Joachim Neubüser, the "father" of GAP. Special thanks are due to Thomas Breuer, who frequently helped us when we had questions or problems with the system and who also carefully read an early version of the manuscript suggesting a large number of improvements. We also would like to thank the participants of the Representation Theory courses one of the authors taught at the University of Arizona for pointing out mistakes in preliminary versions of the manuscript.

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Frequently used symbols

$\mathrm{Aut}(G)$	group of automorphisms of G
$\mathrm{cf}(G,K)$	K-vector space of class functions on G
$\mathbf{C}_G(g)$	centralizer of g in G
$\mathbb{F}_q$	finite field with q elements
$g \in_G H$	$g \in H^x$ for some $x \in G$
$G = N.H$	extension of N by H, thus $N \trianglelefteq G$ and $G/N \cong H$
$G' = [G,G]$	commutator subgroup of G
$\mathrm{GL}_n(K)$	group of invertible elements in $K^{n\times n}$, $\mathrm{GL}_n(q) := \mathrm{GL}_n(\mathbb{F}_q)$
$H \leq G$	H is a subgroup of G
$H_1 =_G H_2$	$H_1 = H_2^g$ for some $g \in G$
$H_1 \leq_G H_2$	$H_1 \leq H_2^g$ for some $g \in G$
H^g, ${}^gH$	$H^g := g^{-1}Hg$, ${}^gH := gHg^{-1}$ for $H \leq G$ and $g \in G$
id_V	identity map from V to V
$\mathbf{I}_n$, $\mathbf{0}_n$	$n\times n$-identity matrix ($\mathbf{I}_n := [\delta_{i,j}]_{1\leq i,j\leq n}$),$n\times n$-zero matrix
$\mathrm{Irr}_K(G)$	irreducible characters of G over K, $\mathrm{Irr}(G) := \mathrm{Irr}_{\mathbb{C}}(G)$
$\mathrm{IBr}_p(G)$	irreducible p-Brauer characters of G
$K^{n\times n}$	ring of $n\times n$-matrices over a commutative ring K
KG	group algebra of the group G over a commutative ring K
$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$	natural numbers, natural numbers with 0, and integers
$N \rtimes H$	$(= N : H)$ split extension (semidirect product) of N by H
$N \cdot H$	non-split extension of N by H
$\mathbf{N}_G(H)$	normalizer of H in G
$\mathbf{1}_G$	trivial character of G
1_K or 1	one of the commutative ring K.
$\mathrm{Out}(G)$	group of outer automorphisms of a group G
$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	rational, real and complex numbers
$R^\times$	multiplicative group of units (= invertible elements) of a ring R
S_n, A_n	symmetric and alternating group of degree n
$\mathrm{SL}_n(K) :=$	$\{g \in \mathrm{GL}_n(K) \mid \det g = 1\}$, $\mathrm{SL}_n(q) := \mathrm{SL}_n(\mathbb{F}_q)$
$\mathbf{Z}(G)$, $\mathbf{Z}(A)$	the center of a group G or a ring A
$\delta_{i,j}$	Kronecker delta
$\boldsymbol{\zeta}_m$, $\mathbb{Q}_m$	$\boldsymbol{\zeta}_m := \exp(\frac{2\pi i}{m}) \in \mathbb{C}$, $\mathbb{Q}_m := \mathbb{Q}(\boldsymbol{\zeta}_m)$ for $m \in \mathbb{N}$
φ^{T}, A^{T}	transposed linear map or matrix
$(\chi,\psi)_G :=$	$\frac{1}{\lvert G\rvert}\sum_{g\in G}\chi(g)\psi(g^{-1})$ for $\chi,\psi \in \mathrm{cf}(G,K)$