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Introduction to relativistic quantum mechanics

1.1 Tensor notation

In this book, we will most often use so-called *natural units*, which means that we have set $c = 1$ and $\hbar = 1$. Furthermore, a general 4-vector will be written in terms of its contravariant index, i.e.

$$A = (A^\mu) = (A^0, \mathbf{A}), \tag{1.1}$$

where A^0 is the time component and the 3-vector $\mathbf{A}$ contains the three spatial components such that $\mathbf{A} = (A^i) = (A^1, A^2, A^3)$.[1] Thus, the contravariant components A^1, A^2, and A^3 are the *physical* components, i.e. A_x, A_y, and A_z, respectively, whereas the covariant components A_1, A_2, and A_3 will be related to the contravariant components.[2] Specifically, the 4-position vector (or spacetime point) is given by

$$x = (x^\mu) = (x^0, \mathbf{x}) = (x^0, x^1, x^2, x^3) = (ct, x, y, z) = \{c = 1\} = (t, x, y, z). \tag{1.2}$$

Note the 'abuse of notation', which means that we will use the symbol x for both representing the 4-position vector as well as its first spatial component. In addition, we introduce the metric tensor as

$$g = (g_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1), \tag{1.3}$$

which is called the *Minkowski metric*. In this case, the inverse of the metric tensor is trivially given by

$$g^{-1} = (g^{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1). \tag{1.4}$$

[1] Note that we will use the convention that Greek indices take the values 0, 1, 2, or 3, whereas Latin (or Roman) indices take the values 1, 2, or 3.

[2] In order for a vector to be invariant under transformations of coordinate systems, the components of the vector have to contra-vary (i.e. vary in the 'opposite' direction) with a change of basis to compensate for that change. Therefore, vectors (e.g. position, velocity, and acceleration) are contravariant, whereas so-called *dual vectors* (e.g. gradients) are covariant.

Thus, for the Minkowski metric, we have that $g_{\mu\nu} = g^{\mu\nu}$, i.e. the covariant and contravariant components are equal to each other, which does not hold for a general metric. Owing to the choice of the Minkowski metric, it also holds for the 4-vector in Eq. (1.1) that $A^0 = A_0$, $A^1 = -A_1 = A_x$, $A^2 = -A_2 = A_y$, and $A^3 = -A_3 = A_z$. In fact, in order to raise and lower indices of vectors and tensors, we can use the Minkowski metric tensor and its inverse in the following way:

$$A_\mu = g_{\mu\nu} A^\nu, \quad A^\mu = g^{\mu\nu} A_\nu, \tag{1.5}$$

$$T_\mu{}^\nu = g_{\mu\lambda} T^{\lambda\nu}, \quad T_{\mu\nu} = g_{\mu\lambda} g_{\nu\omega} T^{\lambda\omega}, \quad T^\mu{}_\nu = g^{\mu\lambda} T_{\lambda\nu}, \quad T^{\mu\nu} = g^{\mu\lambda} g^{\nu\omega} T_{\lambda\omega}. \tag{1.6}$$

Normally, in tensor notation à la Einstein,[3] upper indices (or superscripts) of vectors and tensors are called contravariant indices, whereas lower indices (or subscripts) are called covariant indices. In addition, note that in Eqs. (1.5) and (1.6), we have used the so-called *Einstein summation convention*, which means that when an index appears twice in a single term, once as an upper index and once as a lower index, it implies that all its possible values are to be summed over.

Using the Minkowski metric, we can also introduce the inner product between two 4-vectors A and B such that

$$g(A, B) = A^T g B = A \cdot B = A^\mu g_{\mu\nu} B^\nu = A^\mu B_\mu = A^0 B_0 + A^i B_i = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}, \tag{1.7}$$

which is **not** positive definite.[4] Therefore, the 'length' of a 4-vector A is given by

$$A^2 = A \cdot A = (A^0)^2 - \mathbf{A}^2, \tag{1.8}$$

where we have again used an abuse of notation, since the symbol A^2 denotes both the second spatial contravariant component of the vector A and the 'length' of the vector A. Nevertheless, note that the 'length' is indefinite, i.e. it can be either positive or negative. One says that A is *time-like* if $A^2 > 0$, *light-like* if $A^2 = 0$, and *space-like* if $A^2 < 0$. In particular, it holds for a 4-position vector x that

$$x^2 = (x^0)^2 - \mathbf{x}^2 = t^2 - x^2 - y^2 - z^2. \tag{1.9}$$

Next, we introduce the Minkowski spacetime M such that $M = (\mathbb{R}^4, g)$, which is the set of all 4-position vectors [cf. Eq. (1.2)]. Note that the metric tensor g is a bilinear form $g : M \times M \to \mathbb{R}$ such that $g(x, y) = g_{\mu\nu} x^\mu y^\nu$, where $g_{\mu\nu}$ are strictly the components of the metric g, which are usually identified with the tensor itself.

[3] In 1921, A. Einstein was awarded the Nobel Prize in physics 'for his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect'.
[4] In general, note that the superscript T denotes the transpose of a matrix.

Finally, we introduce the totally antisymmetric *Levi-Civita (pseudo)tensor* in three spatial dimensions

$$\epsilon^{ijk} = \epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ are even permutations of 1,2,3} \\ -1 & \text{if } ijk \text{ are odd permutations of 1,2,3} \\ 0 & \text{if any two indices are equal} \end{cases} \tag{1.10}$$

as well as in four spacetime dimensions

$$\epsilon^{\mu\nu\lambda\omega} = \begin{cases} 1 & \text{if } \mu\nu\lambda\omega \text{ are even permutations of 0,1,2,3} \\ -1 & \text{if } \mu\nu\lambda\omega \text{ are odd permutations of 0,1,2,3} \\ 0 & \text{if any two indices are equal} \end{cases}, \tag{1.11}$$

which we define such that $\epsilon^{0123} = 1$, which implies that $\epsilon_{0123} = -1$. In addition, in three dimensions, the following contractions hold for the Levi-Civita tensor:

$$\epsilon^{ijk}\epsilon^{ijk} = 6, \tag{1.12}$$

$$\epsilon^{ijk}\epsilon^{k\ell m} = \delta^{i\ell}\delta^{jm} - \delta^{im}\delta^{j\ell}, \tag{1.13}$$

$$\begin{aligned} \epsilon^{ijk}\epsilon^{\ell mn} &= \delta^{i\ell}\left(\delta^{jm}\delta^{kn} - \delta^{jn}\delta^{km}\right) - \delta^{im}\left(\delta^{j\ell}\delta^{kn} - \delta^{jn}\delta^{k\ell}\right) \\ &\quad + \delta^{in}\left(\delta^{j\ell}\delta^{km} - \delta^{jm}\delta^{k\ell}\right), \end{aligned} \tag{1.14}$$

where $\delta^{ij} = \delta_{ij}$ is the *Kronecker delta* such that $\delta^{ij} = 1$ if $i = j$ and $\delta^{ij} = 0$ if $i \neq j$, whereas, in four dimensions, the corresponding relations are:

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} = -24, \tag{1.15}$$

$$\epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\nu} = -6\delta^{\mu}_{\nu}, \tag{1.16}$$

$$\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\lambda\omega} = -2\left(\delta^{\mu}_{\lambda}\delta^{\nu}_{\omega} - \delta^{\mu}_{\omega}\delta^{\nu}_{\lambda}\right), \tag{1.17}$$

$$\epsilon^{\alpha\mu\nu\lambda}\epsilon_{\alpha\omega\rho\sigma} = -\delta^{\mu}_{\omega}\delta^{\nu}_{\rho}\delta^{\lambda}_{\sigma} + \delta^{\mu}_{\rho}\delta^{\nu}_{\omega}\delta^{\lambda}_{\sigma} + \delta^{\mu}_{\omega}\delta^{\nu}_{\sigma}\delta^{\lambda}_{\rho} - \delta^{\mu}_{\rho}\delta^{\nu}_{\sigma}\delta^{\lambda}_{\omega} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\omega}\delta^{\lambda}_{\rho} + \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}\delta^{\lambda}_{\omega}, \tag{1.18}$$

where $\delta^{\mu}_{\nu} \equiv g^{\mu}{}_{\nu} = g_{\nu}{}^{\mu}$ such that $\delta^{\mu}_{\nu} = 1$ if $\mu = \nu$ and $\delta^{\mu}_{\nu} = 0$ if $\mu \neq \nu$.

1.2 The Lorentz group

A *Lorentz transformation* Λ is a linear mapping of M onto itself, $\Lambda : M \to M$, $x \mapsto x' = \Lambda x$,[5] which preserves the inner product, i.e. the inner product is *invariant* under Lorentz transformations:

$$x' \cdot y' = x \cdot y, \quad \text{where } x' = \Lambda x \text{ and } y' = \Lambda y. \tag{1.19}$$

[5] We will denote the set of Lorentz transformations by $\mathcal{L}$, which is called the *Lorentz group*. See Appendix A for the definition of a group as well as a short general discussion on group theory.

In component form, we have $x'^{\mu} = (\Lambda x)^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ and $y'_{\mu} = (\Lambda y)_{\mu} = \Lambda_{\mu}{}^{\lambda} y_{\lambda} = \Lambda_{\mu\lambda} y^{\lambda}$, which means that

$$\Lambda^{\mu}{}_{\nu} \Lambda_{\mu\lambda} = g_{\nu\lambda} \quad \Leftrightarrow \quad (\Lambda^{T})_{\nu}{}^{\mu} g_{\mu\omega} \Lambda^{\omega}{}_{\lambda} = g_{\nu\lambda}. \tag{1.20}$$

Thus, the Lorentz group is given by

$$\mathcal{L} = \{\Lambda : M \to M : x' \cdot y' = x \cdot y, \quad \text{where } x' = \Lambda x,\ y' = \Lambda y, \text{ and } x, y \in M\}, \tag{1.21}$$

i.e. it consists of real, linear transformations that leave the inner product invariant, and hence, one says that the Lorentz group is an *invariance group*.

An explicit example of a Lorentz transformation relating two *inertial frames* (or *inertial coordinate systems*)[6] S and S' (with 4-position vectors x and x', respectively) that move along the positive x^1-axis is given by

$$x' = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \Lambda^{(01)} x, \tag{1.22}$$

where ξ is any real number and $\Lambda^{(01)} = \Lambda^{(01)}(\xi)$ denotes the Lorentz transformation. Note that $\Lambda^{(01)}$ is often called the *standard configuration Lorentz transformation* and constitutes an example of a *boost (or standard transformation)*. The parameter ξ is called the *rapidity (or boost parameter)*. Using the hyperbolic identity $\cosh^2\xi - \sinh^2\xi = 1$, one easily observes that indeed $x'^2 = x^2$. By direct computation, one finds that $\Lambda^{(01)}(\xi + \xi') = \Lambda^{(01)}(\xi)\Lambda^{(01)}(\xi')$. Similar to Eq. (1.22), one could define the Lorentz transformations $\Lambda^{(02)}$ and $\Lambda^{(03)}$. Another way of writing the Lorentz transformation $\Lambda^{(01)}$ is

$$\Lambda^{(01)} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.23}$$

where the two parameters β and γ are introduced as

$$\beta \equiv \frac{v}{c} = \{c = 1\} = v, \tag{1.24}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-v^2}} = \gamma(v). \tag{1.25}$$

[6] An inertial frame is a reference system in which free particles (i.e. particles that are **not** influenced by any forces) are moving with uniform velocity. Any reference system moving with constant velocity relative to an inertial frame is another inertial frame. Note that there are infinitely many inertial frames and that the laws of physics have to have the same form in all inertial frames, i.e. the laws of physics are so-called *Lorentz covariant*.

Here v is the relative velocity of the two inertial frames. Note that, comparing Eqs. (1.22) and (1.23), it holds that

$$\cosh\xi = \gamma \quad \text{and} \quad \sinh\xi = \beta\gamma = v\gamma. \tag{1.26}$$

Thus, it follows that the rapidity is given by

$$\xi = \operatorname{artanh} v \quad \Leftrightarrow \quad \tanh\xi = v. \tag{1.27}$$

The physical interpretation is as follows. A particle (or an observer K) at rest in the inertial frame S is represented by a *world-line* parallel to the time axis, i.e. the x^0-axis. Without loss of generality, let us assume that K is at rest at the origin of the three-dimensional space in S. The same K can also be viewed from another inertial frame S', which is related to S by the Lorentz transformation $\Lambda^{(01)}$. In the S'-coordinates, the world-line of K is given by

$$x'^0 = \tau\cosh\xi, \quad x'^1 = -\tau\sinh\xi, \quad x'^2 = x'^3 = 0,$$

where we have used $x^0 = \tau$ and $x^1 = x^2 = x^3 = 0$ for the S-coordinates. The velocity of K along the x'^1-axis is now

$$v' = \frac{\mathrm{d}x'^1}{\mathrm{d}t'} = -\tanh\xi \leq 0,$$

whereas the velocity along the other axes is zero. Thus, we can interpret this result as either (i) K (or the particle) is moving with velocity $-v'$ along the negative x'^1-axis or (ii) S' is moving with velocity $v = \mathrm{d}x^1/\mathrm{d}t = \tanh\xi \geq 0$ along the positive x^1-axis (see Fig. 1.1). Note that $v' = -v$.

Now, using the inner product $g(\Lambda x, \Lambda y) = \Lambda x \cdot \Lambda y = x \cdot y = g(x, y)$, where $x, y \in M$ and $\Lambda \in \mathcal{L}$, we find that

(1) $x \cdot y = x^T g y$ and
(2) $\Lambda x \cdot \Lambda y = (\Lambda x)^T g(\Lambda y) = x^T \Lambda^T g \Lambda y$.

If conditions (1) and (2) are equivalent, which they are, since the inner product is invariant under Lorentz transformations, they imply that

$$g = \Lambda^T g \Lambda, \tag{1.28}$$

which is basically the same equation as Eq. (1.20), but in matrix form. From this equation, one obtains

- $(\det\Lambda)^2 = 1 \quad \Rightarrow \quad \det\Lambda = \pm 1$ and
- $\left(\Lambda^0{}_0\right)^2 = 1 + \sum_{i=1}^3 \left(\Lambda^i{}_0\right)^2 \geq 1 \quad \Rightarrow \quad \Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$.

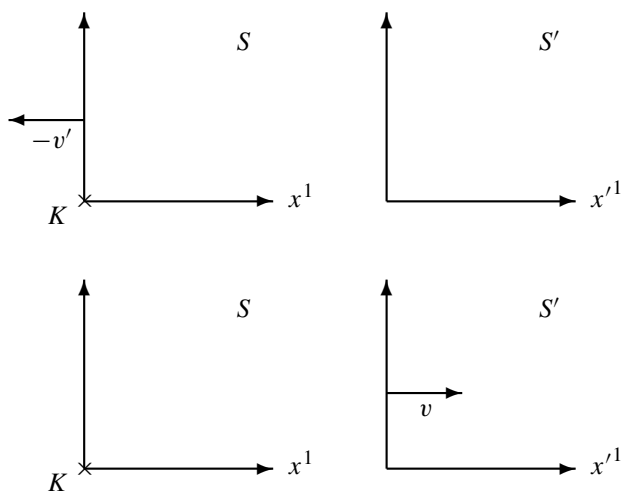

Figure 1.1 The two upper inertial frames show the first interpretation, i.e. S is moving to the left relative to S', whereas the two lower inertial frames show the second interpretation, i.e. S' is moving to the right relative to S.

The four conditions $\det \Lambda = \pm 1$, ${\Lambda^0}_0 \geq 1$, and ${\Lambda^0}_0 \leq 1$ can be used to classify the elements of the Lorentz group. Thus, we can divide the Lorentz group $\mathcal{L}$ [denoted SO(1, 3) in group theory] into the following subgroups (see Fig. 1.2):

$$\mathcal{L}_+ = \{\Lambda \in \mathcal{L} : \det \Lambda = 1\} \qquad \text{pure (or proper) Lorentz group,} \tag{1.29}$$

$$\mathcal{L}^\uparrow = \left\{\Lambda \in \mathcal{L} : {\Lambda^0}_0 \geq 1\right\} \qquad \text{orthochronous Lorentz group,} \tag{1.30}$$

$$\mathcal{L}_+^\uparrow = \mathcal{L}_+ \cap \mathcal{L}^\uparrow \qquad \text{pure and orthochronous Lorentz group.} \tag{1.31}$$

Note that the three subsets

$$\mathcal{L}_-^\uparrow = \left\{\Lambda \in \mathcal{L} : \det \Lambda = -1,\, {\Lambda^0}_0 \geq 1\right\}, \tag{1.32}$$

$$\mathcal{L}_-^\downarrow = \left\{\Lambda \in \mathcal{L} : \det \Lambda = -1,\, {\Lambda^0}_0 \leq 1\right\}, \tag{1.33}$$

$$\mathcal{L}_+^\downarrow = \left\{\Lambda \in \mathcal{L} : \det \Lambda = 1,\, {\Lambda^0}_0 \leq 1\right\} \tag{1.34}$$

are **not** subgroups of $\mathcal{L}$. However, $\mathcal{L}_0 = \mathcal{L}_+^\uparrow \cup \mathcal{L}_-^\downarrow$ is a subgroup of $\mathcal{L}$. Hence, $\mathcal{L}_+$, $\mathcal{L}^\uparrow$, $\mathcal{L}_+^\uparrow$, and $\mathcal{L}_0$ are the invariant subgroups of $\mathcal{L}$. The other subsets of $\mathcal{L}$, which are not subgroups, can be connected to $\mathcal{L}_+^\uparrow$ by the three corresponding and following Lorentz transformations.

(1) *Parity* (or *space inversion*).

$$\Lambda_P = \mathrm{diag}(1, -1, -1, -1) \in \mathcal{L}_-^\uparrow \tag{1.35}$$

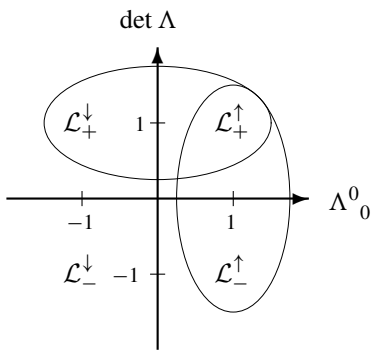

Figure 1.2 Subgroups of the Lorentz group $\mathcal{L}$. The intersection of the two ellipses corresponds to the pure and orthochronous Lorentz group $\mathcal{L}_+^\uparrow$, whereas the horizontal and vertical ellipses correspond to the subgroups $\mathcal{L}_+$ and $\mathcal{L}^\uparrow$, respectively.

(2) *Time reversal* (or *time inversion*).

$$\Lambda_T = \mathrm{diag}(-1, 1, 1, 1) \in \mathcal{L}_-^\downarrow \tag{1.36}$$

(3) *Spacetime inversion*.

$$\Lambda_{PT} = \Lambda_P \Lambda_T = \mathrm{diag}(-1, -1, -1, -1) \in \mathcal{L}_+^\downarrow \tag{1.37}$$

Thus, it holds that $\mathcal{L}_-^\uparrow = \Lambda_P \mathcal{L}_+^\uparrow$, $\mathcal{L}_-^\downarrow = \Lambda_T \mathcal{L}_+^\uparrow$, and $\mathcal{L}_+^\downarrow = \Lambda_{PT} \mathcal{L}_+^\uparrow$, which are so-called *cosets* of $\mathcal{L}$ with respect to $\mathcal{L}_+^\uparrow$. The four subsets (one subgroup and three cosets) $\mathcal{L}_+^\uparrow$, $\mathcal{L}_-^\uparrow$, $\mathcal{L}_-^\downarrow$, and $\mathcal{L}_+^\downarrow$ are disjoint and **not** continuously connected (see again Fig. 1.2). Finally, we obtain

$$\mathcal{L} = \mathcal{L}_+^\uparrow \cup \mathcal{L}_-^\uparrow \cup \mathcal{L}_-^\downarrow \cup \mathcal{L}_+^\downarrow = \mathcal{L}_+^\uparrow \cup \Lambda_P \mathcal{L}_+^\uparrow \cup \Lambda_T \mathcal{L}_+^\uparrow \cup \Lambda_{PT} \mathcal{L}_+^\uparrow. \tag{1.38}$$

The pure and orthochronous (or restricted) Lorentz group $\mathcal{L}_+^\uparrow$ [denoted $\mathrm{SO}^+(1, 3)$ in group theory] is a matrix Lie group (see Appendix A.2), which means that every matrix can be written in the form $\exp\left(-\frac{\mathrm{i}}{2}\omega_{\mu\nu} M^{\mu\nu}\right)$, where $\omega_{\mu\nu} \in \mathbb{R}$ such that $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and $M^{\mu\nu}$ are the so-called *generators* of $\mathcal{L}_+^\uparrow$. The number of generators of $\mathcal{L}_+^\uparrow$ is six, since we define $M^{\mu\nu} = -M^{\nu\mu}$, which implies that $M^{00} = M^{11} = M^{22} = M^{33} = 0$. Thus, $\mathcal{L}_+^\uparrow$ is a six-parameter group. For example, the infinitesimal generator for the 'rotation' in the x^0x^1-plane corresponding to the Lorentz transformation $\Lambda^{(01)}$ is given by

$$M^{01} = \mathrm{i}\, \frac{\mathrm{d}\Lambda^{(01)}(\xi)}{\mathrm{d}\xi}\bigg|_{\xi=0} = \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.39}$$

The other five infinitesimal generators M^{02}, M^{03}, M^{32}, M^{13}, and M^{21} can be derived in a similar way. Next, we introduce the infinitesimal generators J^i ($i = 1, 2, 3$) and K^i ($i = 1, 2, 3$) for $\mathcal{L}_+^\uparrow$, corresponding to rotations and boosts (or standard transformations), respectively. Note that J^i and K^i are constructed in terms of $M^{\mu\nu}$, or vice versa, which we will investigate more in Sections 1.3 and 1.4. Nevertheless, it holds that

$$J^i = -\frac{1}{2}\epsilon^{ijk}M^{jk} \quad \text{and} \quad K^i = M^{0i}. \tag{1.40}$$

Then, one finds by simple computations that

$$\left[J^i, J^j\right] = \mathrm{i}\epsilon^{ijk}J^k, \tag{1.41}$$
$$\left[J^i, K^j\right] = \mathrm{i}\epsilon^{ijk}K^k, \tag{1.42}$$
$$\left[K^i, K^j\right] = -\mathrm{i}\epsilon^{ijk}J^k, \tag{1.43}$$

which form a Lie algebra (see Appendix A.3). Actually, defining the operators

$$\mathbf{j} = \frac{1}{2}\left(\mathbf{J} + \mathrm{i}\mathbf{K}\right) \quad \text{and} \quad \mathbf{k} = \frac{1}{2}\left(\mathbf{J} - \mathrm{i}\mathbf{K}\right), \tag{1.44}$$

i.e. linear combinations of the operators $\mathbf{J}$ and $\mathbf{K}$, one obtains the following commutation relations

$$\left[j^i, j^j\right] = \mathrm{i}\epsilon^{ijk}j^k, \tag{1.45}$$
$$\left[j^i, k^j\right] = 0, \tag{1.46}$$
$$\left[k^i, k^j\right] = \mathrm{i}\epsilon^{ijk}k^k, \tag{1.47}$$

which give an alternative basis for the Lie algebra. From the commutation relations in Eqs. (1.45)–(1.47), we observe that the operators $\mathbf{j}$ and $\mathbf{k}$ are decoupled. This is described by $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ called the Lorentz algebra,[7] which is the Lie algebra of the Lie group SU(2) $\otimes$ SU(2) that can be represented as $D^j \otimes D^{j'}$, where D^j $\left(j = 0, \frac{1}{2}, 1, \ldots\right)$ is an irreducible representation of SU(2). Note that D^j is spanned by basis vectors $|j, m\rangle$, where $m = -j, -j+1, \ldots, j$. Thus, we have the basis vectors $|j, m; j', m'\rangle = |j, m\rangle|j', m'\rangle$. In addition, we find the relations

$$\mathbf{J}^2 - \mathbf{K}^2 = 2\left(\mathbf{j}^2 + \mathbf{k}^2\right), \tag{1.48}$$
$$\mathbf{J}\cdot\mathbf{K} = -\mathrm{i}\left(\mathbf{j}^2 - \mathbf{k}^2\right), \tag{1.49}$$

which are invariant forms (i.e. they commute with the generators $\mathbf{j}$ and $\mathbf{k}$) that are multiplets of the unit operator $\mathbb{1}$ with the eigenvalues $2[j(j+1) + j'(j'+1)]$ and $-\mathrm{i}[j(j+1) - j'(j'+1)]$, respectively. In fact, the invariant forms can be written as

[7] Note that $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$, but the Lie group generated by $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$ is SU(2) and **not** SO(3).

$$\mathbf{J}^2 - \mathbf{K}^2 = \frac{1}{2} M^{\mu\nu} M_{\mu\nu}, \tag{1.50}$$

$$\mathbf{J} \cdot \mathbf{K} = \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} M_{\mu\nu} M_{\rho\sigma}. \tag{1.51}$$

The irreducible representations are denoted by $D^{(j,j')} \equiv D^j \otimes D^{j'}$, where $j, j' = 0, \frac{1}{2}, 1, \ldots$ For example, an explicit representation for the generators of $D^{\left(\frac{1}{2},0\right)}$ and $D^{\left(0,\frac{1}{2}\right)}$ is given by

$$\mathbf{j} = \frac{1}{2}\boldsymbol{\sigma},\ \mathbf{k} = \mathbf{0} \quad \text{and} \quad \mathbf{j} = \mathbf{0},\ \mathbf{k} = \frac{1}{2}\boldsymbol{\sigma}, \tag{1.52}$$

respectively, where $\boldsymbol{\sigma}$ is the vector of *Pauli matrices*, i.e.

$$\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \tag{1.53}$$

which satisfy the commutation relation $[\sigma^i, \sigma^j] = 2\mathrm{i}\epsilon^{ijk}\sigma^k$. Thus, this leads to

$$D^{\left(\frac{1}{2},0\right)}: \quad \mathbf{J} = \frac{1}{2}\boldsymbol{\sigma}, \quad \mathbf{K} = -\mathrm{i}\frac{1}{2}\boldsymbol{\sigma}, \tag{1.54}$$

$$D^{\left(0,\frac{1}{2}\right)}: \quad \mathbf{J} = \frac{1}{2}\boldsymbol{\sigma}, \quad \mathbf{K} = \mathrm{i}\frac{1}{2}\boldsymbol{\sigma}. \tag{1.55}$$

1.3 The Poincaré group

The *Poincaré group* (or the inhomogeneous Lorentz group), which is a (linear) Lie group, is given by

$$\mathcal{P} = \left\{ (\Lambda, a) : x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \right\}, \tag{1.56}$$

where $\Lambda \in \mathcal{L}$ is a Lorentz transformation and $a \in \mathbb{R}^4$ is a translation. Thus, the Lorentz group and the translation group are subgroups of the Poincaré group. In addition, note that one has $\mathcal{P}_+^\uparrow$ if $\Lambda \in \mathcal{L}_+^\uparrow$, i.e. the pure and orthochronous Poincaré group. The group multiplication law of the Poincaré group is

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2), \tag{1.57}$$

where (Λ_1, a_1) and (Λ_2, a_2) are two elements of the Poincaré group, i.e. $\Lambda_1, \Lambda_2 \in \mathcal{L}$ and $a_1, a_2 \in \mathbb{R}^4$. In addition, the identity element is $(\mathbb{1}_4, 0)$ and the inverse is given by $(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a)$, where $(\Lambda, a) \in \mathcal{P}$. An element of the Poincaré group (Λ, a) can be represented by a 5×5 matrix in the following way:

$$(\Lambda, a) \mapsto \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix}, \tag{1.58}$$

which means that the group multiplication law (1.57) simply corresponds to ordinary matrix multiplication of 5×5 matrices. The generators of the Poincaré group are $M^{\mu\nu}$ and P^{μ}, which give rise to the unitary operators that represent the elements $(\Lambda, 0)$ and $(\mathbb{1}_4, a)$ of the Poincaré group, i.e.

$$U(\Lambda, 0) = \exp\left(-\tfrac{\mathrm{i}}{2}\omega_{\mu\nu}M^{\mu\nu}\right), \tag{1.59}$$

$$U(\mathbb{1}_4, a) = \exp(\mathrm{i}a_{\mu}P^{\mu}), \tag{1.60}$$

where again $\omega_{\mu\nu}$ and $M^{\mu\nu}$ are the parameters and generators of the Lorentz subgroup, respectively, and a_{μ} and P^{μ} are the parameters and generators of the translation subgroup, respectively. Note that if the elements of the Poincaré group are close to the identity or to first order in infinitesimal parameters of the Poincaré group, we have

$$U(\Lambda, a) \simeq \exp\left(-\tfrac{\mathrm{i}}{2}\omega_{\mu\nu}M^{\mu\nu} + \mathrm{i}a_{\mu}P^{\mu}\right). \tag{1.61}$$

The different generators of the Poincaré group satisfy the following commutation relations:

$$[M^{\mu\nu}, M^{\rho\sigma}] = \mathrm{i}(g^{\mu\rho}M^{\nu\sigma} + g^{\nu\sigma}M^{\mu\rho} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho}), \tag{1.62}$$

$$[M^{\mu\nu}, P^{\sigma}] = -\mathrm{i}(g^{\nu\sigma}P^{\mu} - g^{\mu\sigma}P^{\nu}), \tag{1.63}$$

$$[P^{\mu}, P^{\nu}] = 0, \tag{1.64}$$

which are the commutation relations of the *Poincaré algebra*, i.e. the algebra that corresponds to the Poincaré group. Finally, returning to the example of the infinitesimal generator for the 'rotation' in the x^0x^1-plane, i.e. M^{01} in Eq. (1.39), the corresponding generator of the Poincaré group is given by

$$M^{01} = \mathrm{i}\left.\frac{\mathrm{d}(\Lambda^{(01)}(\xi), 0)}{\mathrm{d}\xi}\right|_{\xi=0} = \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.65}$$

In the case of the Lorentz group, the generator M^{01} is represented by a 4×4 matrix, whereas in the case of the Poincaré group, the generator M^{01} is represented by a 5×5 matrix.

Exercise 1.1 Verify the commutation relations $[J^1, J^2] = \mathrm{i}J^3$, $[K^1, K^2] = -\mathrm{i}J^3$, and $[J^1, K^2] = \mathrm{i}K^3$ using Eq. (1.62) as well as the definitions $\mathbf{J} = (J^1, J^2, J^3) = (M^{32}, M^{13}, M^{21})$ and $\mathbf{K} = (K^1, K^2, K^3) = (M^{01}, M^{02}, M^{03})$.