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978-0-521-76542-8 - Stochastic Processes for Physicists: Understanding Noisy Systems

Kurt Jacobs

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STOCHASTIC PROCESSES FOR PHYSICISTS

Understanding Noisy Systems

Stochastic processes are an essential part of numerous branches of physics, as well as biology, chemistry, and finance. This textbook provides a solid understanding of stochastic processes and stochastic calculus in physics, without the need for measure theory.

In avoiding measure theory, this textbook gives readers the tools necessary to use stochastic methods in research with a minimum of mathematical background. Coverage of the more exotic Levy processes is included, as is a concise account of numerical methods for simulating stochastic systems driven by Gaussian noise. The book concludes with a non-technical introduction to the concepts and jargon of measure-theoretic probability theory.

With over 70 exercises, this textbook is an easily accessible introduction to stochastic processes and their applications, as well as methods for numerical simulation, for graduate students and researchers in physics.

KURT JACOBS is an Assistant Professor in the Physics Department at the University of Massachusetts, Boston. He is a leading expert in the theory of quantum feedback control and the measurement and control of quantum nano-electro-mechanical systems.

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To Salman Habib and Bala Sundaram,
for pointing the way.

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Preface

This book is intended for a one-semester graduate course on stochastic methods. It is specifically targeted at students and researchers who wish to understand and apply stochastic methods to problems in the natural sciences, and to do so without learning the technical details of measure theory. For those who want to familiarize themselves with the concepts and jargon of the “modern” measure-theoretic formulation of probability theory, these are described in the final chapter. The purpose of this final chapter is to provide the interested reader with the jargon necessary to read research articles that use the modern formalism. This can be useful even if one does not require this formalism in one’s own research.

This book contains more material than I cover in my current graduate class on the subject at UMass Boston. One can select from the text various optional paths depending on the purpose of the class. For a graduate class for physics students who will be using stochastic methods in their research work, whether in physics or interdisciplinary applications, I would suggest the following: Chapters 1, 2, 3 (with Section 3.8.5 optional), 4 (with Section 4.2 optional, as alternative methods are given in 7.7), 5 (with Section 5.2 optional), 7 (with Sections 7.8 and 7.9 optional), and 8 (with Section 8.9 optional). In the above outline I have left out Chapters 6, 9 and 10. Chapter 6 covers numerical methods for solving equations with Gaussian noise, and is the sort of thing that can be picked-up at a later point by a student if needed for research. Chapter 9 covers Levy stochastic processes, including exotic noise processes that generate probability densities with infinite variance. While this chapter is no more difficult than the preceding chapters, it is a more specialized subject in the sense that relatively few students are likely to need it in their research work. Chapter 10, as mentioned above, covers the concepts and jargon of the rigorous measure-theoretic formulation of probability theory.

A brief overview of this book is as follows: Chapters 1 (probability theory without measure theory) and 2 (ordinary differential equations) give background material that is essential for understanding the rest of course. Chapter 2 will be almost

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all revision for students with an undergraduate physics degree. Chapter 3 covers all the basics of Ito calculus and solving stochastic differential equations. Chapter 4 introduces some further concepts such as auto-correlation functions, power spectra and white noise. Chapter 5 contains two applications (Brownian motion and option pricing), as well as a discussion of the Stratonovich formulation of stochastic equations and its role in modeling multiplicative noise. Chapter 6 covers numerical methods for solving stochastic equations. Chapter 7 covers Fokker–Planck equations. This chapter also includes applications to reaction–diffusion systems, and pattern formation in these systems. Chapter 8 explains jump processes and how they are described using master equations. It also contains applications to population dynamics and neuron behavior. Chapter 9 covers Levy processes. These include noise processes that generate probability densities with infinite variance, such as the Cauchy distribution. Finally Chapter 10 introduces the concepts and jargon of the “modern” measure-theoretic description of probability theory.

While I have corrected many errors that found their way into the manuscript, it is unlikely that I eliminated them all. For the purposes of future editions I would certainly be grateful if you can let me know of any errors you find.

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Aric Hagberg for providing me with a beautiful plot of labyrinthine pattern formation in reaction–diffusion systems (see Chapter 7) and Jason Ralph for bringing to my attention Edwin Jaynes’ discussion of mathematical style (see Chapter 10). I am also grateful to my students for being patient and helping me iron-out errors and omissions in the text.