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The impact of QFT on low-dimensional topology

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Abstract

In this chapter I discuss some of the history and problems of geometric topology and how ideas coming from physics have had an impact on low-dimensional topology in the last 20 years. The ideas are presented largely in simplified (and morally but not necessarily rigorously correct) form to give students an overview of the topics unencumbered by the many technical issues required to put the results on a firm theoretical footing.

The goal of this chapter is to provide theoretical physics students with an introduction to the impact of modern physics on mathematics, as well as to provide for mathematics students a gentle but broad introduction to some of the developments in topology inspired by quantum field theory. No prerequisites are needed besides the usual mathematical maturity, but the astute student will recognize the large role that Morse theory plays. Thus, some familiarity with Morse theory is likely to be useful.

1.1 Geometric topology: a brief history

Geometric topology refers to the study of (usually compact) manifolds.

Let $\mathbb{R}^n_{\geq} = \{(x_1, \ldots, x_n) | x_n \geq 0\}$. An *n-dimensional manifold* is a topological space M equipped with a maximal collection of *charts*

$$\{(U_i, \phi_i) \,|U_i \subset M \text{ open}, \phi_i : U_i \to \mathbb{R}^n_{\geq}, \quad \phi_i \text{ a homeomorphism onto an open subset}\}.$$

The set of points mapped to $\{x_n = 0\}$ by the charts ϕ_i is called the *boundary of M*, denoted ∂M. If M is compact and ∂M is empty, we call M *closed*.

There are many different notions of manifold. Manifolds can have many different kinds of extra structures or restrictions (and corresponding equivalences),

* Dedicated to the memory of my friend K. P. Wojciechowski.

such as an orientation (orientation-preserving homeomorphism), a smooth structure (diffeomorphism), a PL structure (PL isomorphism), a spin structure (spin diffeomorphism), a $Spin^c$ structure, an almost complex structure, a symplectic structure (symplectomorphism), a Riemannian or Lorentzian structure (isometry), a flat or spherical or hyperbolic structure, a holomorphic structure (biholomorphism), a Kähler structure, a framing, a trivial fundamental group, a contractible universal cover, etc.

In geometric topology the focus is on *structures such that the corresponding set of equivalence classes is discrete*, and the goal of geometric topology can usually be stated as follows:

Distinguish all equivalence classes of manifolds with a given structure.

1.1.1 Examples

To a compact, connected 2-manifold one can associate its *Euler characteristic* χ (alternating sum of numbers of n-simplices in a triangulation), the number b of boundary circles, and $o \in \{0, 1\}$ keeping track of whether or not the manifold is orientable. Then a classical theorem of topology states that *two compact 2-manifolds have the same triple* (χ, b, o) *if and only if they are homeomorphic*. Thus the class of compact, connected 2-manifolds is classified up to homeomorphism by $(\chi, b, o) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}/2$, and it is simple to determine which triples occur.

Another example is provided by a consequence of Smale's h-cobordism theorem [48]: *Every closed manifold homotopy equivalent to an n-sphere* $S^n = \{x \in \mathbb{R}^{n+1} | \|x\| = 1\}$ *is homeomorphic to an n-sphere, if* $n > 4$. (Freedman [15] proved this for $n = 4$, and Perelman for $n = 3$.)

Two topological spaces X, Y are *homotopy equivalent* if there exists continuous maps $f : X \to Y$ and $g : Y \to X$ and homotopies $H : X \times [0, 1] \to X$ and $K : Y \times I \to Y$ such that $H(x, 0) = x$, $H(x, 1) = g(f(x))$, $K(y, 0) = y$, $K(y, 1) = f(g(y))$.

More interestingly, two *smooth* closed n-manifolds homotopy-equivalent to S^n need not be diffeomorphic. But a consequence of Smale's theorem is that if M^n is a smooth homotopy sphere, $n \geq 5$, then M is obtained from a pair of hemispheres D^n_+ and D^n_- (with $D^n_\pm = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$) by gluing their boundaries using a nontrivial diffeomorphism $f : \partial D^n_+ \cong \partial D^n_-$.

Gluing, or *pasting*, topological spaces X, Y along subsets $A \subset X$ and $B \subset Y$ using a *gluing map* $f : A \to B$ refers to forming the quotient space $X \cup Y/\sim$, where $x \sim y$ if $x \in A$, $y \in B$, and $f(x) = y$. Gluing n-manifolds using a homeomorphism f of their boundaries results in an n-manifold. If M is an n-manifold and $N \subset M$ is an $(n-1)$-submanifold with $\partial N = N \cap \partial M$, then *cutting* M *along* N means forming the manifold (with nonempty boundary) obtained by taking the closure of $M - N$. There are technical issues to worry about, and often one uses instead the complement $M - nbd(N)$ of a small tubular neighborhood of N. Notice that gluing and cutting are inverse operations.

An earlier example is provided by Thom's cobordism theorem [53]: *two closed manifolds M, N are cobordant if and only if they have the same Stiefel–Whitney numbers.* Thom determined which numbers occur as Stiefel–Whitney numbers.

Closed n manifolds M and N are *cobordant* if there exists a compact $n+1$-manifold W with ∂W the disjoint union of M and N. The *Stiefel–Whitney numbers* of a manifold are a collection of numbers $w_I \in \mathbb{Z}/2$; one for each multi-index $I = (i_1, i_2, \ldots, i_n)$, $i_1 + 2i_2 + \cdots + ni_n = n$.

Yet another example is given by Freedman's theorem [15]: *two simply connected (see the following) closed 4-manifolds are homeomorphic if and only if they have isomorphic cohomology rings (see the next section for an introduction to cohomology) and the same Kirby–Seibenmann invariant $KS \in \mathbb{Z}/2$.* One new twist here is that there remains one unsolved case of the classification of possible ring structures; namely, the full classification of unimodular quadratic forms over $\mathbb{Z}$ (which is determined by and determines the cohomology ring of a simply connected 4-manifold) is not known.

Thus one might say that the homeomorphism classification of simply connected 4-manifolds is *reduced* to an algebra problem. In the case of Thom's theorem, Thom first reduced the cobordism classification to a homotopy theory problem, then he solved the homotopy theory problem. For Freedman's theorem, the classification reduces to an algebra problem which is largely solved, but not completely. This is typical.

As a negative example, it is simple to prove that any finitely presented group is the fundamental group of a closed n-manifold for any $n \geq 4$. Logicians tell us the problem of determining whether two group presentations give isomorphic groups is not solvable (no algorithm exists to determine if two presentations determine isomorphic groups). Thus there cannot be an algorithmic (e.g., a finite set of invariants) homeomorphism classification of n-manifolds for $n \geq 4$.

The *fundamental group* $\pi_1(X, x)$ of a topological space X with a distinguished base point is the set of based homotopy classes of loops $\alpha : [0, 1] \to X, \alpha(0) = \alpha(1) = x$. So $\alpha \sim \beta$ if there is a map $H : [0, 1] \times [0, 1] \to X$ so that $H(t, 0) = \alpha(t)$, $H(t, 1) = \beta(t)$, $H(0, u) = x = H(1, u)$. The group structure is given by following one loop, then the next. Concisely, it is the group of path components of the based loop space on X. A continuous map $f : X \to Y$ induces a homomorphism $\pi_1(X) \to \pi_1(Y)$. A connected space X is called *simply connected* if $\pi_1(X, x)$ is the trivial group.

A good 3-dimensional theory is provided by Waldhausen's results on Haken 3-manifolds [56]. A closed, oriented 3-manifold M is called *Haken* if it contains a closed oriented 2-manifold $F \subset M$ (with $F \neq S^2$) such that $\pi_1(F) \to \pi_1(M)$ is injective and such that every sphere $S^2 \subset M$ cuts M into two pieces, one of which is D^3. Then Waldhausen's theorem says: *if M and N are Haken and $\pi_1(M)$ is isomorphic to $\pi_1(N)$, then M and N are homeomorphic (and diffeomorphic).* Thus the fundamental group "classifies" Haken manifolds. Many (but not all) 3-manifolds are Haken, or can be cut into Haken pieces which can be analyzed. Later Thurston proved [54] that *a closed Haken 3-manifold which does not contain a π_1-injective* torus *admits a hyperbolic structure.* Thus this class of 3-manifolds is classified by identifying its fundamental group with a Kleinian group.

An important family of examples comes from considering pairs (M, N) where M is an m-dimensional manifold and N is an n-dimensional submanifold (with $n < m$). One can ask for a *relative* homeomorphism classification, i.e., assuming M_0 is homeomorphic to M_1 and N_0 is homeomorphic to N_1, does there exist a homeomorphism $h : M_0 \cong M_1$ such that $h(N_0) = N_1$? Other interesting questions include the *concordance problem*, where one assumes $M_0 = M_1 = M$, and sets $(M, N_0) \sim (M, N_1)$ if there exists an embedding of $N \times [0, 1] \subset M \times [0, 1]$ which restricts to (M, N_0) and (M, N_1) at the ends $M \times \{0\}$ and $M \times \{1\}$.

The most interesting case is when $n = m - 2$, the *codimension 2 embedding problem*. This topic is generally known as *knot theory*, especially when $M = S^m$ and $N \cong S^{m-2}$. The further specialization when $n = 3$, i.e., the study of embeddings $S^1 \subset S^3$, is usually called *classical knot theory* (and was first systematically studied by the physicist Lord Kelvin, who theorized that tiny knots in the "æther" might explain the subatomic properties of nature).

1.1.2 Invariants

The preceding examples show that classifying manifolds in a certain class is a subtle problem. One should not expect as clean an answer as for 2-manifolds, or even Thom's cobordism theorem. There are several questions to consider: What class of manifolds do we study? Under what equivalence relation? What kind of methods can we use? What is considered progress?

The standard approach to a classification problem in geometric topology is to

(i) find a geometrically meaningful and computable set of *invariants*,
(ii) find a collection of manifolds in the class that *realize* all the invariants (or determine their range),
(iii) prove these invariants classify.

The term *invariants* refers to some way of associating some object (e.g., in a category, or set, or group) to each manifold in the chosen class so that equivalent manifolds are given the same invariant. "Geometrically meaningful" is a vague term, but ideally the invariants should tell us something interesting about the geometric problem. As a negative example, consider the invariant of closed n-manifolds I defined by $I(M) = 0$ if M is homeomorphic to a sphere and $I(M) = 1$ otherwise. Then I is an invariant which partially classifies n manifolds, but it is useless, because its definition reveals nothing about the underlying geometric question.

By contrast, the Euler characteristic of a closed, orientable 2-manifold classifies up to homeomorphism, but in addition it can be defined for any compact space, and it can be computed in many ways (e.g., from a triangulation, by computing homology, from a Morse function, or geometrically by the Gauss–Bonnet theorem). Moreover it has many nice properties (multiplicativity under covers, independence from the triangulation, etc.). Thus producing new invariants is not by itself progress (despite frequent claims made to the contrary).

An important requirement is that invariants be computable. This is also a vague requirement, but to the extent that there are cut-and-paste constructions to produce new manifolds from old in the given class, a good interpretation of this requirement is that it is desirable to be able to compute how the invariant changes under specific cut-and-paste operations.

1.1.3 High and low dimensions

Geometric topology is divided into two distinct topics: *high-dimensional topology*, i.e., the study of manifolds of dimension 5 and higher, and *low-dimensional topology*, i.e., manifolds of dimensions 2, 3, and 4. The reason for this dichotomy is technical, but boils down to the slogan "there is more room to move in high dimensions." A beautiful construction due to Whitney [59], called the *Whitney trick*, uses 2-dimensional disks as guides for various geometric deformations. In n-dimensional topology with $n > 4$, because $2 + 2 < n$ it follows that it is easy to fit the 2-dimensional disk into a manifold in such a way that it does not interfere with itself and other disks in the manifold (in the same way that circles in 3-space can be moved off each other by arbitrarily small perturbations, because $1 + 1 < 3$).

The upshot of this is that the Whitney trick allows one to prove injectivity of invariants produced by counting intersections of submanifolds in high dimensions. This was exploited in the golden era of geometric topology (1955–1980) by many mathematicians, including Smale, Milnor, Wall, Browder, Sullivan, and Novikov, by combining cutting and pasting constructions (surgery) with related algebra (algebraic K-theory) and homotopy theory (bordism theory). These techniques are combined in a powerful machine called *surgery theory*.

It is not correct to say that all classification problems of high-dimensional topology are solved, but it is fair to say that many important ones have been, and that surgery theory provides a starting point for investigating high-dimensional problems which is often powerful enough to solve the problem, or at least to reduce it to purely algebraic or discrete problems.

In low-dimensional topology, the situation is less clear. In dimension 4, there is a chasm between the topological theory and the smooth theory (i.e., between classification up to homeomorphism and up to diffeomorphism). The homeomorphism problem was treated satisfactorily by Freedman. The result is that many of the techniques of surgery theory extend to dimension 4, albeit with much more intricate proofs using infinite processes in point-set topology. Interestingly, the homotopy classification of non-simply-connected 4-manifolds is not well understood. Freedman's results treat the gap between the homotopy problem and the homeomorphism problem for many fundamental groups.

The diffeomorphism problem is quite different, and a breakthrough came at about the same time as Freedman's theorem when Donaldson used ideas from physics (gauge theory) to produce invariants of differentiable 4-manifolds [5]. The development of the ideas pioneered by Donaldson has been the focus of most of the work in the last 20 years in 4-dimensional topology. A major simplification came a few years later with the introduction of Seiberg–Witten theory [28, 47]. However, it is fair to say that any kind of diffeomorphism classification in dimension 4 is a distant goal. As R. Stern puts it, in smooth 4-dimensional topology, "the more we learn the more we realize how little we know."

In 3-dimensional topology (and 2-dimensional topology) there is no difference between homeomorphism and diffeomorphism questions. Depending on one's point of view (and the time of day), 3-manifolds are either well understood or mysterious. One feature of 3-dimensional topology is that there are many structure theorems, notably the existence and uniqueness of decompositions along 2-spheres (the *connected sum* decomposition theorem [30]) and then along tori (the *Jaco–Shalen–Johannsen torus* decomposition theorem [22, 23]). The results of Waldhausen have already been mentioned: these build on many previous results, but notably on Papakyriakopoulos's proofs of Dehn's lemma and the sphere theorem [42].

The next major step forward occurred when Thurston proved his hyperbolization theorem, which inserted the beautiful techniques of Kleinian groups into the study of 3-manifolds. The recent stunning results of Perelman on Thurston's geometrization conjecture can be considered as a continuation of this perspective in 3-dimensional topology.

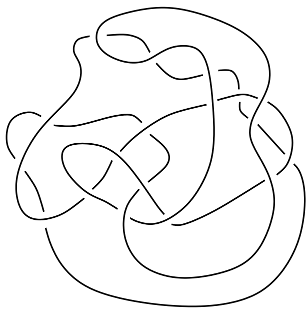

Fig. 1.1. A projection of the unknotted circle

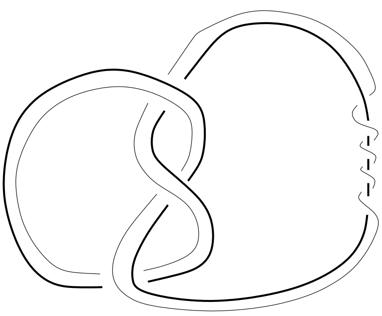

Fig. 1.2. A framed trefoil knot

1.1.4 Links, Reidemeister moves, and Kirby's theorem

We introduce a few notions from the theory of knots and links, a subject that is both of intrinsic interest in topology and also a useful tool in the construction of manifolds.

A *link* in S^3 is an embedding of a finite disjoint union of circles in S^3,

$$\bigsqcup_{i=1}^{n} S^1_i \subset S^3.$$

A link with one path component (i.e., $n = 1$) is called a *knot*.

A *projection* of a knot or link is a picture of a generically immersed curve in $\mathbb{R}^2$, with "over and under" data given at each double point, to specify a knot or link in $\mathbb{R}^3 = S^3 - \{p\}$. See Figure 1.1.

A *framed* link in S^3 is an embedding of a finite disjoint union of solid tori in S^3,

$$\bigsqcup_{i=1}^{n} (S^1 \times D^2)_i \subset S^3.$$

See Figure 1.2.

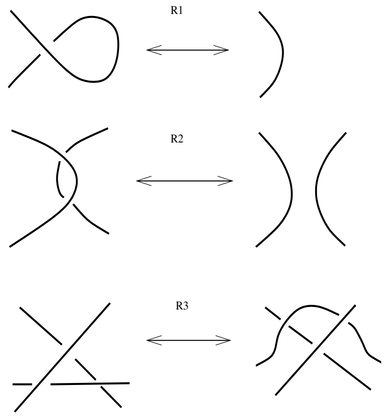

Fig. 1.3. The three Reidemeister moves

The result of *surgery on a framed link* is the 3-manifold, M, obtained by cutting out each $(S^1 \times D^2)$ and gluing in $(D^2 \times S^1)$:

$$M = \left(S^3 - \bigsqcup_i (S^1 \times D^2)_i \right) \cup \bigsqcup_i (D^2 \times S^1)_i .$$

Isotopy of links can be described using projections in terms of the three *Reidemeister moves*, illustrated in Figure 1.3. In fact, Reidemeister proved that these three moves classify links in S^3, in the sense that two *projections* (i.e., pictures like those in Figures 1.1 and 1.2) of links correspond to equivalent links if and only if one can get from one projection to the other by a sequence of Reidemeister moves R1, R2, and R3.

In a different direction, a classification theorem of sorts for 3-manifolds was proven by Kirby in [26].

It has been known since the 1950s that any 3-dimensional manifold is obtained by surgery on a framed link in S^3. However, many different framed links yield the same manifold.

There are geometric moves on the set of framed links: isotopy, stabilization (adding a small, appropriately framed knot away from the rest, as in Figure 1.4), and addition, or sliding (adding a parallel copy of one component to another component; see Figure 1.5).

Kirby's theorem says *two framed links give diffeomorphic 3-manifolds if and only if the framed links are related by these moves*. (The stabilization and sliding moves in this dimension are often called *Kirby moves*.) In other words, 3-manifolds are classified by identifying them with equivalence classes of framed links in S^3 (note that any framed link can be visualized as circles in $\mathbb{R}^3$ with numbers attached to them).

This suggests a strategy to approach the classification problem for 3-manifolds: construct a function which assigns a complex number (or an element in an abelian

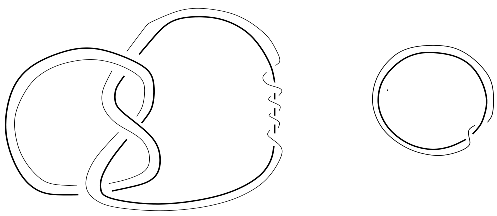

Fig. 1.4. A (-1) stabilization of a framed trefoil knot

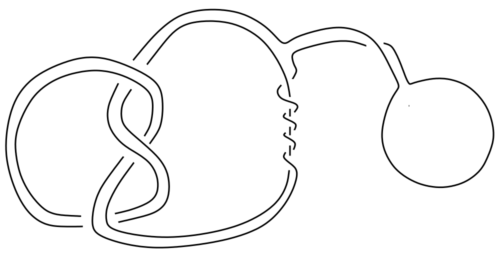

Fig. 1.5. Sliding one component over another (framings omitted)

group) to each link in S^3 (respectively, framed link in S^3) in such a way that links related by Reidemeister moves are assigned the same number (respectively, framed links related by Reidemeister and Kirby moves are assigned the same number). This might be a good strategy because one can draw links and framed links. It may be hard to prove directly that two link projections correspond to different equivalence classes, but straightforward to show that certain functions are preserved by Reidemeister or Kirby moves.

This is one place where QFT has had an impact on low-dimensional topology: The use of Feynman path integrals and the strategy by which physicists compute them have led to the invention of the new mathematical notion of TQFT. Perhaps more importantly, the input from physics has led to the formulation of a set of axioms similar to, but in an important sense fundamentally different from, the axioms of a homology theory.

1.2 Homology theories

We first review axioms for cohomology theories, which reshaped mathematical thinking in the second half of the twentieth century. We will take the point of view suited to algebraic topology, but there are many other points of view, in geometry, algebra, analysis, etc. For reasons of exposition we discuss cohomology theories, but each cohomology theory corresponds to a unique homology theory.

A *category* $\mathcal{C}$ is a class of *objects* $\mathrm{Ob}(\mathcal{C})$, together with a class of disjoint sets $\mathrm{Hom}(A, B)$, called *morphisms*, one for each pair $A, B \in \mathrm{Ob}(\mathcal{C})$. Moreover, we require that for each triple A, B, C of objects there exist a *composition* $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ denoted $(f, g) \mapsto g \circ f$ satisfying

(i) *(Associativity)* $(f \circ g) \circ h = f \circ (g \circ h)$,
(ii) *(Identity)* for each $A \in \mathrm{Ob}(\mathcal{C})$ there exists a $1_A \in \mathrm{Hom}(A, A)$ such that for each B, we have $1_A \circ f = f$ for $f \in \mathrm{Hom}(B, A)$ and $g \circ 1_A = g$ for $g \in \mathrm{Hom}(A, B)$.

A *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ is one that assigns to each object $A \in \mathrm{Ob}(\mathcal{C})$ an object $F(A) \in \mathrm{Ob}(\mathcal{D})$ and to each morphism $h \in \mathrm{Hom}(A, B)$ a morphism $F(h) \in \mathrm{Hom}(F(A), F(B))$ so that compositions and identity are preserved. A *contravariant functor* is defined in a similar way, except that if $h \in \mathrm{Hom}(A, B)$, then $F(h) \in \mathrm{Hom}(F(B), F(A))$, i.e., the arrows are reversed.

A *natural transformation* between two functors is, loosely speaking, a functor of functors. More precisely, if $F, G : \mathcal{C} \to \mathcal{D}$ are two covariant functors, then a natural transformation $n : F \to G$ assigns to each $A \in \mathrm{Ob}(\mathcal{C})$ a morphism $n(A) \in \mathrm{Hom}(F(A), G(A))$ such that if $f \in \mathrm{Hom}(A, B)$ is a morphism in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "n(A)"'] & F(B) \arrow[d, "n(B)"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

commutes. A similar definition works for contravariant functors.

Definition 1.2.1 Let $\mathcal{T}$ denote the category of topological spaces with base points (or, to be safe, CW complexes) and continuous maps, and $\mathcal{A}$ the category of (graded) abelian groups. Let $S : \mathcal{T} \to \mathcal{T}$ be the *suspension functor*, i.e., $SX = X \times [0, 1]/\sim$, where $X \times \{0, 1\} \cup \{p\} \times [0, 1]$ is collapsed to a point.

A (reduced) *cohomology theory* is a contravariant functor $h : \mathcal{T} \to \mathcal{A}$ together with a degree 1 natural transformation $e : h \circ S \to h$ satisfying the following axioms:

(i) *(Homotopy)* If $f_0, f_1 : X \to Y$ are (based) homotopic, then
$$h(f_0) = h(f_1) : h(Y) \to h(X).$$
(ii) *(Exactness)* If $X \subset Y$, then the sequence
$$h(X/Y) \to h(Y) \to h(X)$$
is an exact sequence.
(iii) *(Suspension)* For each X, the natural transformation
$$e(X) : h(SX) \to h(X)$$
is an isomorphism.