This book on personal financial planning and wealth management employs the lifecycle model of financial economics. The central idea of “consumption smoothing” is used to connect chapters and topics such as saving and investment, debt management, risk management, and retirement planning. The first part of the book is nontechnical and aimed at a wide audience with no special technical background. The second part of the book provides a rigorous presentation of the lifecycle model from first principles using the calculus of variations. The accompanying website is found at http://www.yorku.ca/milevsky/?page_id=185.

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Strategic Financial Planning over the Lifecycle

A Conceptual Approach to Personal Risk Management

NARAT CHARUPAT
McMaster University, Canada

HUAXIONG HUANG
York University, Canada

MOSHE A. MILEVSKY
York University and the IFID Centre, Canada
To our wives...
As far as I am aware, no one has challenged the view that if people were capable of it, they ought to plan their consumption, saving, and retirement according to the principles enunciated by Modigliani and Brumberg in the 1950s.

Angus S. Deaton, Princeton University, 2005
Contents

Acknowledgments  xiii

1 Introduction and Motivation 1

2 Mathematical Preliminaries – Working with Interest Rates 4

  2.1 Interest Rates 4
      2.1.1 Effective Annual Rate (EAR) and Annual Percentage Rate (APR) 5

  2.2 Time Value of Money – Definitions 7

  2.3 Present Value and Future Value of a Single Cash Flow 8

  2.4 Present Value of an Annuity with Constant Payments 9
      2.4.1 Ordinary Annuity 9
      2.4.2 Annuity Due 11

  2.5 Future Value of an Annuity with Constant Payments 12
      2.5.1 Ordinary Annuity 12
      2.5.2 Annuity Due 13

  2.6 Present Value of an Annuity Whose Payments Change at a Constant Rate 14
      2.6.1 Constant-Growth Ordinary Annuity 14
      2.6.2 Constant-Growth Annuity Due 18

  2.7 Future Value of an Annuity Whose Payments Change at a Constant Rate 19
      2.7.1 Constant-Growth Ordinary Annuity 19
      2.7.2 Constant-Growth Annuity Due 20

  2.8 A Preliminary Discussion of the Effects of Inflation 21

  2.9 Appendix: Geometric Progressions and Annuity Relations 23

  2.10 Questions and Assignments 26
3 Personal Balance Sheet and Human Capital 28
  3.1 Personal Balance Sheets 28
     3.1.1 Financial Assets 30
     3.1.2 Gross Human Capital 31
     3.1.3 Explicit Liabilities 32
     3.1.4 Implicit Liabilities 32
     3.1.5 Economic Net Worth 32
  3.2 Estimation of Gross Human Capital 33
     3.2.1 What Is Proper Discount Rate for Gross Human Capital? 36
  3.3 Estimation of Implicit Liabilities 37
  3.4 An Example of Human Capital Decisions: Investing in Education 40
     3.4.1 Some Caveats 42
  3.5 Questions and Assignments 42

4 Consumption Smoothing and Optimal Savings 46
  4.1 Consumption Smoothing 46
  4.2 Assumptions for Consumption Smoothing 49
  4.3 Solving for Smoothness 51
     4.3.1 Effects of Changes in Other Parameter Values 65
  4.4 Consumption Smoothing vs. Other Approaches 70
     4.4.1 The Fixed-Percentage Savings Rules 70
     4.4.2 The 70% Income Replacement Rule 73
  4.5 Practical Considerations 75
  4.6 Questions and Assignments 76

5 Debts, Loans, and Mortgages [Canadian Content] 79
  5.1 Mortgage Financing 79
     5.1.1 Mortgage Amortization 81
     5.1.2 Mortgage Insurance 84
     5.1.3 Mortgage Refinancing 88
     5.1.4 Fixed-Rate vs. Floating-Rate Loans 92
  5.2 Borrowing to Invest 95
  5.3 Debt Consolidation 98
  5.4 Is Debt Good or Bad? 99
  5.5 Questions and Assignments 100

6 Personal Income Taxes [Canadian Content] 103
  6.1 The Logic of Income Tax Calculations 103
Contents

6.1.1 Federal Income Taxes 105
6.1.2 Provincial Income Taxes 109
6.1.3 Combining Federal and Provincial Income Taxes 111
6.2 Investment Income 114
6.2.1 Interest Income 115
6.2.2 Dividend Income 115
6.2.3 Capital Gains/Losses 119
6.3 Income Tax Planning 120
6.3.1 Income Splitting 121
6.3.2 Tax-Deferred Investment Vehicles 124
6.3.3 Tax-Free Savings Accounts (TFSAs) 131
6.3.4 RRSPs vs. TFSAs 132
6.4 Questions and Assignments 134

7 Risk, Utility, and Insurance 136
7.1 The Concepts of Risk and Risk Preferences 136
7.1.1 Definition of Risk 136
7.1.2 Risk Preferences 137
7.2 The Concept of Insurance 138
7.3 The Demand for Insurance 141
7.3.1 When Should People Buy Insurance? 141
7.3.2 How Much Should You Pay for Insurance? 142
7.3.3 The Effect of Current Wealth on Insurance Decisions 147
7.3.4 The Effect of Risk-Aversion Parameter on Insurance Decisions 149
7.3.5 Generalizations 151
7.4 Insurance and Consumption Smoothing 152
7.5 Questions and Assignments 155

8 Mortality Risk and Life Insurance 157
8.1 Who Needs Life Insurance? 157
8.2 How Much Coverage Do You Need? 158
8.2.1 Human-Capital Approach 158
8.2.2 The Expense (Needs) Approach 160
8.3 Types of Life Insurance 161
8.3.1 Term Life Insurance 162
8.3.2 Cash-Value Life Insurance 179
8.4 Questions and Assignments 182
## Contents

9 Investment and Diversification 183
   9.1 Investment Decisions and Consumption 183
   9.2 Major Asset Classes 186
      9.2.1 Cash Equivalents 186
      9.2.2 Fixed Income (Bonds) 187
      9.2.3 Equities 188
      9.2.4 Commodities 189
      9.2.5 Real Estate 190
   9.3 Historical Returns 191
      9.3.1 Arithmetic Average vs. Geometric Average 193
      9.3.2 Correlations of Returns 194
   9.4 Asset Allocation and Diversification 194
   9.5 Mutual Funds and Exchange-Traded Funds 197
      9.5.1 Mutual Funds 197
      9.5.2 Exchange-Traded Funds (ETFs) 201
   9.6 Summary 201
   9.7 Questions and Assignments 202

10 The Mathematics of Portfolio Diversification 204
   10.1 Impressive Investment Performance: Real or Fake 204
   10.2 Portfolio Diversification 206
      10.2.1 Preliminaries 206
      10.2.2 From Bernoulli to Binomial 209
      10.2.3 The Central Limit Theorem 211
   10.3 Portfolio Construction 214
      10.3.1 Efficient Frontier for Two Assets 217
      10.3.2 Efficient Frontier for More Than Two Assets 219
   10.4 Summary 221
   10.5 Questions and Assignments 222

11 Housing Decisions 224
   11.1 Buying or Renting? 224
      11.1.1 From a Purely Financial Perspective 224
      11.1.2 Other Aspects of the Decision 232
      11.1.3 Our Recommendation 234
   11.2 Questions and Assignments 236

12 Pensions and Retirement [Canadian Content] 238
   12.1 Longevity Risk 238
   12.2 Retirement Plans that Provide Lifelong Income 243
### Contents

12.2.1 Canada Pension Plan (CPP) and Old-Age Security Pension (OAS) .................................................. 243  
12.2.2 Defined-Benefit Pension Plan ............................................................................................................. 244  
12.2.3 Life Annuity ........................................................................................................................................ 248  
12.3 Optimal Retirement Spending Rate in the Presence of Longevity Risk .................................................. 254  
12.3.1 Retirement Spending Rates .............................................................................................................. 255  
12.3.2 Numerical Example ............................................................................................................................ 256  
12.3.3 Including Pension Annuities ............................................................................................................. 258  
12.4 Summary ............................................................................................................................................... 262  
12.5 Questions and Assignments .................................................................................................................. 264  
13 Advanced Material: Part I. Continuous Time and the Calculus of Variations .................................... 266  
13.1 Wages and Salary in Continuous Time .................................................................................................. 266  
13.2 Human Capital Values in Continuous Time ............................................................................................ 267  
13.3 A Different Route: Same Destination ..................................................................................................... 270  
13.4 Does Human Capital Grow or Shrink over Time? .................................................................................. 272  
13.5 Implicit Liabilities in Continuous Time .................................................................................................. 273  
13.6 Consumption Smoothing: A Second Look ............................................................................................. 273  
13.7 Calculus of Variations ............................................................................................................................ 279  
13.8 Smooth Consumption: A Final Look ...................................................................................................... 281  
13.9 Life and Death in Continuous Time ....................................................................................................... 283  
13.10 The Problem of Retirement Income .................................................................................................... 291  
13.10.1 The Euler-Lagrange Equation .......................................................................................................... 293  
13.10.2 Wealth Depletion .............................................................................................................................. 293  
13.10.3 Classifying the Different Retirement Trajectories .......................................................................... 295  
13.10.4 Explicit Solution: Exponential Remaining Lifetime ..................................................................... 297  
13.10.5 Explicit Solution: Gompertz Mortality ............................................................................................ 299  
13.11 Back to the Working Years .................................................................................................................... 302  
13.11.1 Patient Individual $k \leq \hat{k} < g$ ...................................................................................................... 303  
13.11.2 Relatively Impatient Individual $k \leq g \leq \hat{k}$ .................................................................................. 305  
13.11.3 Impatient Individual $g < k \leq \hat{k}$ .................................................................................................. 308  
13.12 Conclusion .......................................................................................................................................... 309  
13.13 Questions and Assignments .................................................................................................................. 310  
13.14 Appendix ............................................................................................................................................ 313
14 Advanced Material: Part II. Stochastic Optimal Control and the HJB Equation 315

14.1 Dynamic Asset Allocation and Optimal Consumption 315
14.2 Asset Price Model 316
14.3 A Self-Financed Portfolio 317
14.4 The Asset-Allocation Problem 318
14.5 The Asset-Allocation-Consumption Problem 321
14.6 The Asset-Allocation-Consumption Problem with Deterministic Wage Income 323
14.7 An Advanced Stochastic Lifecycle Model 324

14.7.1 Model Specifications 325
14.7.2 HJB Equation for the Lifecycle Model 328
14.7.3 Closed-Form Solution 329
14.8 Questions and Assignments 337
14.9 Technical Appendix 338

14.9.1 Elementary Stochastic Calculus in Finance 338
14.9.2 Stochastic Dynamic Optimization 341
14.9.3 Finite Difference Method for Solving the HJB Equation 342

15 Concluding Thoughts and Next Steps 345

15.1 Portfolio Selection with Human Capital 346
15.2 Asset Location vs. Asset Allocation 347
15.3 Home Mortgage Financing 348
15.4 Life and Health Insurance 348
15.5 Employee Compensation and Proper Incentives 349
15.6 Choice of Defined Benefit vs. Defined Contribution Pensions 349
15.7 Starting a Retirement Pension Annuity 350
15.8 Estate Planning and Tax Arbitrage 350
15.9 The Real Options in Your Life 350

Bibliography 353

Index 357
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