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Excerpt
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# ONE

# Introduction and Motivation

This book grew out of the deep frustration two of the three authors have experienced for many years, trying to teach a practical yet rigorous course on personal – in contrast to corporate or investment – finance, to undergraduate and graduate students at Canadian business schools. Although there are many college-level textbooks that discuss the *tactical* aspects of personal finance – nuggets such as: credit card debt is bad; reduce fees on mutual funds; regular saving is important; have a budget; and so on – we have not come across a textbook that integrated all these disparate concepts into a conceptual or *strategic* framework for financial decision making, *based on sound economic principles.*

For the most part, personal finance is being taught as a collection of stand-alone facts about "smart" money management. Most existing textbooks are written assuming a (very) basic background in mathematics on the part of the student, which limits the financial and economic level at which such a course can be delivered and the material discussed. Moreover, in today's Google and YouTube world, curious students could obtain more relevant, accurate, and up-to-date information about most (if not all) of the products that are part of the personal financial toolkit. In our opinion, a textbook that allocates most of its pages to the *tactical* aspects of financial planning, such as explaining how to read a credit card statement or how to get a copy of your credit report from your local credit bureau, or how to open up a brokerage account, is not advanced enough for a third or fourth year course in a business school. Many personal finance textbooks – whose first editions were written before Internet browsers existed – are a relic from a bygone era in which good information could not be located on the Web within seconds.

On a related note – to make things more difficult for us as instructors – the recent "discoveries" by behavioral economists that individuals do not adhere to the most basic axioms of rational choice, and the overwhelming evidence that most consumers make systematic and persistent mistakes with their money, create an even greater need for *a conceptual framework* that goes beyond amusing anecdotes regarding dollar bills left lying on the street.

Likewise, from the perspective of students in the engineering and sciences, it is often difficult for those who are interested in personal finance problems to find a background textbook that provides an overview of the relevant institutional features of personal finance and insurance together with the mathematical treatments used to solve these problems. Often, mathematics students have to sit through various courses offered in business schools, occasionally extracting useful information from a mountain of tangential material.

So, like any authors embarking on an ambitious writing project, we believe there is a niche to be filled. In particular, our goal in writing this book – and the way we approach the topic in class – is to *teach personal finance from the perspective of the lifecycle model* (LCM), originally formulated mathematically by Ramsey (1928), economically by Fisher (1930), then refined by Modigliani and Brumberg (1954) as well as Friedman (1957), and finally adjusted for lifetime uncertainty by Yaari (1965). Our intention is to extract as many practical insights as possible in an accessible and analytically tractable manner. If there is one question that links every single dilemma in personal finance, it is: **What course of action will help me maximize my standard of living – in the smoothest way possible – over the rest of my life?**

To avoid distraction, mathematical techniques are only presented when they are absolutely needed. Our emphasis is on the practical aspects of these techniques rather than mathematical rigor. The first twelve chapters of this book – which are geared toward undergraduates in business and finance – present the lifecycle model of investment and consumption under very simple assumptions about wages, retirement dates, and investment returns. The final two chapters (13 and 14) are (much) more mathematical and present advanced material related to the LCM, leading up to the Merton (1990) work on asset allocation in continuous time. The two final chapters are more suitable for an advanced undergraduate audience in economics and applied mathematics, or perhaps a first year graduate course, assuming they have the mathematical maturity and interest. As far as the numerical examples and case studies are concerned, we focus our examples on the Canadian environment (and in particular the tax material in Chapter 6) mainly because this is where we are located and where we currently teach. The other chapters or sections that contain a substantial amount of Canadian content are so designated. That said, most of the conceptual material – which forms the majority of this book – is universal enough to apply anywhere.

In writing this book we aimed to "prove" that personal finance can be taught to university students in an intellectually satisfying manner, within a rational and strategic framework. We hope you agree.

TWO

# Mathematical Preliminaries – Working with Interest Rates

**Learning Objectives**

In this chapter, we will review the concepts of interest rates and time value of money (TVM). There are several types of present-value and future-value formulas, each of which is used in specific circumstances. Our goal is to make sure that you understand when (i.e., in what context) these formulas should be used. A good understanding of this chapter is needed to proceed to future chapters, where we will need to calculate the amounts of your consumption and savings at various points in time.

Although we believe that most of you have covered these materials in your previous finance courses, we recommend that you take another look at them and familiarize yourself with the notations we will use in the rest of this book.

## 2.1 Interest Rates

As you may recall, an interest rate is the rate of return that a borrower promises to pay for the use of money that he or she borrows from the lender. Normally, it is expressed in terms of per-annum percentage rates (e.g., 4% p.a.). To express it properly, however, we also need to state the compounding frequency of the rate, which is the number of compounding periods in one year. In other words, it is the number of times in a year that interest is calculated and added to the principal of the loan.

For example, annual compounding means that interest is added to the principal once a year. Suppose you invest \$1 for one year at the interest rate of 4% p.a., annual compounding. At the end of the year, you will receive:

$$(1 + 0.04)^1 = 1.04.$$

Suppose instead that the interest rate is 4% p.a., semi-annual compounding. In this case, interest will be added to the principal every six months (i.e., a compounding period is six months). In other words, you earn 2% every six months, with the interest being reinvested. So, after one year you will get:

$$\left(1 + \frac{0.04}{2}\right)^2 = 1.0404,$$

which is slightly more than the annual-compounding case. On the other hand, if the rate is 4% p.a., monthly compounding (i.e., a compounding period is one month), you now earn $0.04/12 = 0.3333\%$ per month, with the interest being reinvested. As a result, you will end up with:

$$\left(1 + \frac{0.04}{12}\right)^{12} = 1.040742,$$

which is even higher.

As you can see, the higher the frequency of compounding, the more money you will receive at the end of your investment horizon. Note from these calculations that what we do in each of the three cases is: (i) we figure out the rate per compounding period based on the stated compounding frequency, and (ii) add that per-period rate to 1 and then raise the whole thing to the power of the number of compounding periods. Formally, if you invest $\$A$ for $n$ years at an interest rate of $i$% p.a., compounded $m$ times per year, at the end of $n$ years, you will receive:

$$A \cdot \left(1 + \frac{i}{m}\right)^{mn}. \tag{2.1}$$

### 2.1.1 Effective Annual Rate (EAR) and Annual Percentage Rate (APR)

Next, we want to introduce a term that reflects the returns that you get under various compounding frequencies. That term is **effective annual rate (EAR)**. An EAR is simply the interest rate that you actually earn after taking compounding into account. For example, when the stated interest rate is 4% p.a., annual compounding, the EAR is also 4% because this is the rate that you actually earn after one year. On the other hand, when the stated interest rate is 4% p.a., semi-annual compounding, the EAR is 4.04%. Finally, when the stated interest rate is 4% p.a., monthly compounding, the EAR is 4.0742%. Given a per-annum interest rate, the higher the frequency of compounding, the higher the EAR will be.

Formally, suppose the stated interest rate is $i$% p.a., compounded $m$ times per year. Its EAR is:

$$\left(1+\frac{i}{m}\right)^{m}-1. \tag{2.2}$$

Obviously, if you know the EAR and the compounding frequency, you can use equation (2.2) to work backward to find the stated interest rate, $i$.

In practice, some lenders express their interest rates in terms of **annual percentage rates (APRs)**. This is common among, for example, car dealers and credit card issuers. An APR is a simple interest rate per year, and is expressed without an associated compounding frequency. A good way to think of an APR is as follows. Suppose the lender wants to charge you some interest rate per payment period (which can be of any length – a month, a quarter, a year, etc.). The APR is calculated by multiplying that rate per payment period by the number of payment periods in one year. For example, suppose you are quoted an interest rate of 4% APR, and the lender requires you to make a payment every month (i.e., payment period is one month). This means that the lender is charging you a rate of 0.3333% per payment period (i.e., a month). This is because if you multiply the rate per period by the number of periods in a year, you will get $0.3333\% \times 12 = 4\%$.

There are two things to note about APRs. First, because an APR quote does not come with an associated compounding frequency, it is up to the borrower to know the number of payment periods per year. Secondly, an APR does not take into account the compounding effect. Consider the previous example where you were quoted an interest rate of 4% APR, payable monthly, which means the rate per payment period is 0.3333%. The effective annual rate (EAR) in this case is:[1]

$$(1.003333)^{12}-1=0.040742 \text{ or } 4.0742\%.$$

In other words, when the payments are required more often than once a year, the effective interest rate that you pay will be higher than the quoted APR. Now you can see that the quoted APR of 4% is not the rate that you effectively end up paying over a year. That effective rate is the EAR, which in this case is 4.0742%. This is why an APR quote is ambiguous. You need to know the payment frequency in order to figure out the effective annual rate.

[1] Note that the EAR in this case is the same as the EAR when the rate is quoted as 4% p.a., monthly compounding. This is because the rate per period in both cases is the same, which is 0.3333% per month.

Of course, given a quoted APR, different frequencies of payments will lead to different EARs.

In the previous example, the difference between the APR (4%) and the EAR (4.0742%) may not appear to be significant. However, with a higher quoted APR, the difference can be substantial. For example, many credit cards charge interest at an APR of 25% (or higher) with daily compounding. At first glance, 25% would translate to $250 on a credit card balance of $1,000, which is already a bit of money. However, that amount has not yet taken into account the compounding effect. What you actually end up paying is the EAR, which in this case is:

$$\left(1 + \frac{0.25}{365}\right)^{365} - 1 = 0.283916 \text{ or } 28.3916\%,$$

which is almost 3.40% more than the stated APR.

Before we move on, we want to emphasize three things from this section. First, it is important to understand the different ways under which interest rates can be expressed (e.g., 4% p.a., semi-annual compounding vs. 4% APR). Secondly, you should be able to distinguish between rates per year and rates per period (where a period can be of any length). Finally, you should be able to work with different compounding frequencies.

## 2.2 Time Value of Money – Definitions

Because money can earn a rate of return, the value of $1 to be received today is not the same as the value of $1 to be received some time in the future (say, one year from now). Therefore, in order to compare properly cash flows that occur at different points in time, we need to apply the concept of time value of money (TVM) so that those cash flows are valued as of one common time point. To this end, we define **present value** as the current worth of an amount of money or a sequence of cash flows. Also, we define **future value** as the value as of a specified date in the future of an amount of money or a sequence of cash flows.

Present value and future value are linked by rates of return. Note that we use the term **rates of return**, which is a more general term than **interest rates**. This is because people can invest their money in many different ways. If they lend it out, they will earn a rate of return equal to the prevailing interest rate. On the other hand, if they invest it some other way (say, buying a stock), they will earn a rate of return that reflects the nature of that investment. Therefore, the use of a more general term is appropriate.

In addition, note that from now on, we also refer to a rate of return as a **valuation rate**.

## 2.3 Present Value and Future Value of a Single Cash Flow

The present value of \$1 to be received $N$ periods from now (note that a period can be of any length) where the valuation rate is $v$% per period is:

$$\mathbf{PV}\,(v,\ N) := \frac{1}{(1+v)^N} = (1+v)^{-N}. \tag{2.3}$$

For example, suppose the valuation rate is $v = 2\%$ per period. The present value of \$1 to be received five periods from now is:

$$\mathbf{PV}\,(0.02,\ 5) = \frac{1}{(1.02)^5} = 0.9057.$$

Next, let us suppose that you now have \$1. What is the future value of it five periods from now? It must be equal to \$1 compounded over five periods at the valuation rate; that is,

$$\mathbf{FV}\,(0.02,\ 5) = 1 \cdot (1.02)^5 = 1.1041.$$

Formally, the future value of \$1 after $N$ periods from now at the valuation rate of $v$% per period is:

$$\mathbf{FV}\,(v,\ N) := (1+v)^N. \tag{2.4}$$

Note that we have kept the length of a period unspecified and that the valuation rate $v$ is expressed as a rate per period. When you want to calculate present values or future values, it is up to you to find the correct rate per period to use. For example, suppose the length of a period is one month and we are dealing with an interest rate quote of 6% APR. It is now up to you to recognize that because the length of a period is one month, the rate per period (based on your knowledge of what an APR is from the previous section) is $6\%/12 = 0.5\%$ per period.

Just to make sure that you are comfortable with finding the rate per period, let us consider another example.

**Practice Question** *Suppose you are quoted an interest rate of 6% p.a., semi-annual compounding. What is the equivalent rate per month?*

This is not as straightforward as in the earlier APR case. In the APR case, you know that the rate per period is simply the APR divided by the number of periods in a year. Here, the equivalent rate per month must be such that it compounds to the same effective annual rate (EAR) as the EAR of the quoted rate, which is:

$$\left(1 + \frac{0.06}{2}\right)^2 - 1 = 0.0609 \text{ or } 6.09\%.$$

As a result, the equivalent rate per month must be such that:

$$(1 + v)^{12} - 1 = 0.0609.$$

Solving for $v$, we get $v = 0.004939$ or 0.4939% per month.

Next, we will look at the present value and the future value of a sequence of cash flows. A typical example is an annuity. An annuity represents a sequence of payments that are made over many consecutive periods. Those payments can be of an equal amount or can vary. Each payment can occur at either the beginning or the end of a period. If the payment is made at the beginning of a period, that annuity is typically referred to as an **annuity due**. On the other hand, if the payment is made at the end of a period, that annuity is typically referred to as an **ordinary annuity**.

We divide our discussion into different cases depending on (i) whether the annuity is an ordinary annuity or an annuity due, and (ii) whether its payments remain the same or vary.

## 2.4 Present Value of an Annuity with Constant Payments

We start with an annuity whose payments remain the same through time.

### 2.4.1 Ordinary Annuity

Let us consider an ordinary annuity that pays \$1 at the end of a period for $N$ periods (i.e., there are $N$ payments). How do we find the present value of this ordinary annuity? The logic must be the same as in the case of the present value of a single amount. Here, what you can do is find the present value of each of the $N$ payments and then add them up. That is, the present value is equal to:

$$\textbf{PVA}(v, N) := \frac{1}{1+v} + \frac{1}{(1+v)^2} + \frac{1}{(1+v)^3} + \cdots + \frac{1}{(1+v)^N}, \tag{2.5}$$

which, after some manipulation, reduces to:

$$\textbf{PVA}(v, N) = \left[\frac{1 - (1+v)^{-N}}{v}\right]. \tag{2.6}$$

The square bracket on the right-hand side of equation (2.6) is what we call a **present-value-of-annuity factor** or PVA factor, denoted by **PVA**$(v, N)$. It gives you the present value of an annuity per \$1 of payoff per period for $N$ periods. So, if the annuity's payment is some other amount (say, \$$X$) per period, you can simply multiply \$$X$ by the PVA factor to get the present value of that annuity.

**Practice Question** *Suppose that you are earning a salary of \$5,000 per month, payable at the end of the month. What is the present value of your future salary, assuming that you will work for another thirty years and that the valuation rate is 6% APR?*

Here, the payment period is monthly and so the valuation rate per period is $0.06/12 = 0.005$ or 0.5% per month. The number of periods is 360 months. The PVA factor is:

$$\mathbf{PVA}\,(0.005,\,360) = \left[\frac{1-(1.005)^{-360}}{0.005}\right] = 166.7916.$$

As a result, the present value of your lifetime salary is \$5,000 $\cdot$ 166.7916 = \$833,958.

Note how the present value depends on the assumption that 6% APR is the appropriate rate to use to value your future salary. Of course, if we used a higher rate (e.g., 8%), the present value will be lower, and vice versa. We discuss in the next chapter how the appropriate rate can be determined.

**Practice Question** *You have just won a lottery worth \$1,000,000. You are going to put the winnings in a bank account earning 6% p.a., annual compounding. You plan to withdraw \$70,000 from this account at the end of every year. How many years will the money last?*

This is a simple present-value-of-annuity problem. The present value of the yearly withdrawals must equal the amount of the winnings, \$1,000,000. In this case, the payment period is one year. Let $N$ be the number of years that you can make the withdrawals. Then,

$$70{,}000 \cdot \mathbf{PVA}\,(0.06,\,N) = 1{,}000{,}000.$$

Using the formula for PVA, we have

$$70{,}000 \cdot \left[\frac{1-(1.06)^{-N}}{0.06}\right] = 1{,}000{,}000,$$