FIELD THEORIES OF CONDENSED MATTER PHYSICS

Presenting the physics of the most challenging problems in condensed matter using the conceptual framework of quantum field theory, this book is of great interest to physicists in condensed matter and high-energy and string theorists, as well as to mathematicians. Revised and updated, this second edition features new chapters on the renormalization group, the Luttinger liquid, gauge theory, topological fluids, topological insulators, and quantum entanglement.

The book begins with the basic concepts and tools, developing them gradually to bring readers to the issues currently faced at the frontiers of research, such as topological phases of matter, quantum and classical critical phenomena, quantum Hall effects, and superconductors. Other topics covered include one-dimensional strongly correlated systems, quantum ordered and disordered phases, topological structures in condensed matter and in field theory and fractional statistics.

EDUARDO FRADKIN is a Professor in the Department of Physics, University of Illinois at Urbana-Champaign. His research interests are in condensed matter physics; disordered systems, high-temperature superconductors, and electronic liquid-crystal phases of strongly correlated systems; quantum Hall fluids and other topological phases of matter; and quantum field theory in condensed matter.
FIELD THEORIES OF CONDENSED MATTER PHYSICS

SECOND EDITION

EDUARDO FRADKIN
University of Illinois at Urbana-Champaign
### Contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface to the second edition</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Preface to the first edition</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>Field theory and condensed matter physics</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>What has been included in this book (first edition)</td>
<td>2</td>
</tr>
<tr>
<td>1.3</td>
<td>What was left out of the first edition</td>
<td>3</td>
</tr>
<tr>
<td>1.4</td>
<td>What has been included in the second edition</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>The Hubbard model</td>
<td>8</td>
</tr>
<tr>
<td>2.1</td>
<td>Introduction</td>
<td>8</td>
</tr>
<tr>
<td>2.2</td>
<td>Symmetries of the Hubbard model</td>
<td>10</td>
</tr>
<tr>
<td>2.3</td>
<td>The strong-coupling limit</td>
<td>14</td>
</tr>
<tr>
<td>2.4</td>
<td>The weak-coupling limit</td>
<td>17</td>
</tr>
<tr>
<td>2.5</td>
<td>Correlation functions</td>
<td>21</td>
</tr>
<tr>
<td>3</td>
<td>The magnetic instability of the Fermi system</td>
<td>27</td>
</tr>
<tr>
<td>3.1</td>
<td>Mean-field theory</td>
<td>27</td>
</tr>
<tr>
<td>3.2</td>
<td>Path-integral representation of the Hubbard model</td>
<td>39</td>
</tr>
<tr>
<td>3.3</td>
<td>Path integrals and mean-field theory</td>
<td>46</td>
</tr>
<tr>
<td>3.4</td>
<td>Fluctuations: the non-linear sigma model</td>
<td>51</td>
</tr>
<tr>
<td>3.5</td>
<td>The Néel state and the non-linear sigma model</td>
<td>57</td>
</tr>
<tr>
<td>4</td>
<td>The renormalization group and scaling</td>
<td>63</td>
</tr>
<tr>
<td>4.1</td>
<td>Scale invariance</td>
<td>63</td>
</tr>
<tr>
<td>4.2</td>
<td>Examples of fixed points</td>
<td>67</td>
</tr>
<tr>
<td>4.3</td>
<td>Scaling behavior of physical observables</td>
<td>72</td>
</tr>
<tr>
<td>4.4</td>
<td>General consequences of scale invariance</td>
<td>75</td>
</tr>
<tr>
<td>4.5</td>
<td>Perturbative renormalization group about a fixed point</td>
<td>78</td>
</tr>
<tr>
<td>4.6</td>
<td>The Kosterlitz renormalization group</td>
<td>82</td>
</tr>
</tbody>
</table>
# Contents

5 One-dimensional quantum antiferromagnets 90
  5.1 The spin-1/2 Heisenberg chain 90
  5.2 Fermions and the Heisenberg model 100
  5.3 The quantum Ising chain 112
  5.4 Duality 116
  5.5 The quantum Ising chain as a free-Majorana-fermion system 118
  5.6 Abelian bosonization 126
  5.7 Phase diagrams and scaling behavior 141

6 The Luttinger liquid 145
  6.1 One-dimensional Fermi systems 145
  6.2 Dirac fermions and the Luttinger model 149
  6.3 Order parameters of the one-dimensional electron gas 153
  6.4 The Luttinger model: bosonization 155
  6.5 Spin and the Luttinger model 159
  6.6 Scaling and renormalization in the Luttinger model 164
  6.7 Correlation functions of the Luttinger model 169
  6.8 Susceptibilities of the Luttinger model 176

7 Sigma models and topological terms 189
  7.1 Generalized spin chains: the Haldane conjecture 189
  7.2 Path integrals for spin systems: the single-spin problem 190
  7.3 The path integral for many-spin systems 198
  7.4 Quantum ferromagnets 199
  7.5 The effective action for one-dimensional quantum antiferromagnets 202
  7.6 The role of topology 205
  7.7 Quantum fluctuations and the renormalization group 209
  7.8 Asymptotic freedom and Haldane’s conjecture 213
  7.9 Hopf term or no Hopf term? 218
  7.10 The Wess–Zumino–Witten model 227
  7.11 A (brief) introduction to conformal field theory 233
  7.12 The Wess–Zumino–Witten conformal field theory 238
  7.13 Applications of non-abelian bosonization 243

8 Spin-liquid states 251
  8.1 Frustration and disordered spin states 251
  8.2 Valence bonds and disordered spin states 253
  8.3 Spinons, holons, and valence-bond states 261
  8.4 The gauge-field picture of the disordered spin states 263
  8.5 Flux phases, valence-bond crystals, and spin liquids 270
8.6 Is the large-$N$ mean-field theory reliable? 277
8.7 SU(2) gauge invariance and Heisenberg models 279

9 Gauge theory, dimer models, and topological phases 286
  9.1 Fluctuations of valence bonds: quantum-dimer models 286
  9.2 Bipartite lattices: valence-bond order and quantum criticality 290
  9.3 Non-bipartite lattices: topological phases 291
  9.4 Generalized quantum-dimer models 292
  9.5 Quantum dimers and gauge theories 294
  9.6 The Ising gauge theory 298
  9.7 The $\mathbb{Z}_2$ confining phase 301
  9.8 The Ising deconfining phase: the $\mathbb{Z}_2$ topological fluid 305
  9.9 Boundary conditions and topology 309
  9.10 Generalized $\mathbb{Z}_2$ gauge theory: matter fields 314
  9.11 Compact quantum electrodynamics 319
  9.12 Deconfinement and topological phases in the $\text{U}(1)$ gauge theory 321
  9.13 Duality transformation and dimer models 325
  9.14 Quantum-dimer models and monopole gases 336
  9.15 The quantum Lifshitz model 342

10 Chiral spin states and anyons 359
  10.1 Chiral spin liquids 359
  10.2 Mean-field theory of chiral spin liquids 366
  10.3 Fluctuations and flux phases 371
  10.4 Chiral spin liquids and Chern–Simons gauge theory 375
  10.5 The statistics of spinons 382
  10.6 Fractional statistics 389
  10.7 Chern–Simons gauge theory: a field theory of anyons 393
  10.8 Periodicity and families of Chern–Simons theories 398
  10.9 Quantization of the global degrees of freedom 400
  10.10 Flux phases and the fractional quantum Hall effect 402
  10.11 Anyons at finite density 405
  10.12 The Jordan–Wigner transformation in two dimensions 412

11 Anyon superconductivity 414
  11.1 Anyon superconductivity 414
  11.2 The functional-integral formulation of the Chern–Simons theory 415
  11.3 Correlation functions 417
  11.4 The semi-classical approximation 418
  11.5 Effective action and topological invariance 424
Contents

12 Topology and the quantum Hall effect 432
  12.1 Quantum mechanics of charged particles in magnetic fields 432
  12.2 The Hofstadter wave functions 438
  12.3 The quantum Hall effect 445
  12.4 The quantum Hall effect and disorder 447
  12.5 Linear-response theory and correlation functions 449
  12.6 The Hall conductance and topological invariance 456
  12.7 Quantized Hall conductance of a non-interacting system 468
  12.8 Quantized Hall conductance of Hofstadter bands 472

13 The fractional quantum Hall effect 480
  13.1 The Laughlin wave function 480
  13.2 Composite particles 499
  13.3 Landau–Ginzburg theory of the fractional quantum Hall effect 502
  13.4 Fermion field theory of the fractional quantum Hall effect 512
  13.5 The semi-classical excitation spectrum 523
  13.6 The electromagnetic response and collective modes 525
  13.7 The Hall conductance and Chern–Simons theory 528
  13.8 Quantum numbers of the quasiparticles: fractional charge 530
  13.9 Quantum numbers of the quasiparticles: fractional statistics 534

14 Topological fluids 536
  14.1 Quantum Hall fluids on a torus 536
  14.2 Hydrodynamic theory 542
  14.3 Hierarchical states 547
  14.4 Multi-component abelian fluids 552
  14.5 Superconductors as topological fluids 556
  14.6 Non-abelian quantum Hall states 563
  14.7 The spin-singlet Halperin states 573
  14.8 Moore–Read states and their generalizations 575
  14.9 Topological superconductors 587
  14.10 Braiding and fusion 597

15 Physics at the edge 603
  15.1 Edge states of integer quantum Hall fluids 603
  15.2 Hydrodynamic theory of the edge states 609
  15.3 Edges of general abelian quantum Hall states 620
  15.4 The bulk–edge correspondence 624
  15.5 Effective-field theory of non-abelian states 641
  15.6 Tunneling conductance at point contacts 647
  15.7 Noise and fractional charge 661
Contents ix

15.8 Quantum interferometers 662
15.9 Topological quantum computation 667

16 Topological insulators 669
16.1 Topological insulators and topological band structures 669
16.2 The integer quantum Hall effect as a topological insulator 670
16.3 The quantum anomalous Hall effect 672
16.4 The quantum spin Hall effect 687
16.5 $\mathbb{Z}_2$ topological invariants 696
16.6 Three-dimensional topological insulators 701
16.7 Solitons in polyacetylene 705
16.8 Edge states in the quantum anomalous Hall effect 714
16.9 Edge states and the quantum spin Hall effect 718
16.10 $\mathbb{Z}_2$ topological insulators and the parity anomaly 720
16.11 Topological insulators and interactions 733
16.12 Topological Mott insulators and nematic phases 736
16.13 Topological insulators and topological phases 745

17 Quantum entanglement 753
17.1 Classical and quantum criticality 753
17.2 Quantum entanglement 756
17.3 Entanglement in quantum field theory 758
17.4 The area law 762
17.5 Entanglement entropy in conformal field theory 765
17.6 Entanglement entropy in the quantum Lifshitz universality class 769
17.7 Entanglement entropy in $\phi^4$ theory 778
17.8 Entanglement entropy and holography 780
17.9 Quantum entanglement and topological phases 785
17.10 Outlook 795

References 799
Index 826
Preface to the second edition

I am extremely happy to, at long last, be able to present the second edition of this book. In spite of what I stated in the preface of the 1991 edition, I ended up not only writing a second edition but, in a sense, a new book. So one can say, once again, that we have met the enemy and it is us. I have been pleased that the 1991 edition of this book was appreciated by many people who found it useful and stimulating. I am really happy that my effort was not in vain.

My motivation for writing this book, in 1991 and now, was to present quantum field theory as a conceptual framework to understand problems in condensed matter physics that cannot be described perturbatively, and hence do not admit a straightforward reduction to some non-interacting problem. In essence, almost all interesting problems in condensed matter physics have this character. Two prime examples of problems of this type in condensed matter physics that developed in the late 1980s, and even more so in the 1990s, are the understanding of high-temperature superconductors and the quantum Hall effects. In both areas field theory played (and plays) a central role. If anything, the use of these ideas has become widespread and increasingly plays a key role. It was lucky that the first edition of this book appeared at just about the right time, even though this meant that I had to miss out on research that was and still is important. This was probably the only time that I was on time, as people who know me can relate. Much has happened since the first edition appeared in print. The problem of the quantum Hall effects has developed into a full-fledged framework to understand topological phases of matter. Although it is still an unsolved problem, the research in high-temperature superconductors (and similar problems) has motivated theorists to look for new ways to think of these problems, and the ideas of quantum field theory have played a central role. The concepts, and subtleties, of gauge theory have come to play a key role in many areas, particularly in frustrated quantum magnetism. The interactions between condensed matter and other areas of physics, particularly high-energy physics and string theory, have become more
important. Concepts in topology and other areas of mathematics rarely frequented by condensed matter physicists have also entered the field with full force. More recent developments have seen the incorporation of ideas of general relativity and quantum entanglement into the field.

These developments motivated me to work on a second edition of this book. I have to thank Simon Capelin, my editor from Cambridge University Press, who took the time to persuade me that this was not a foolish project. So, some time in 2007 (I think) I finally agreed to do it. Of course, this was a more complex project than I had expected (nothing new there!). For this reason it took until now, the Spring of 2012, for me to finish what I thought would take just one year (or so). I wish to thank Simon Capelin and the people at Cambridge University Press for working with me throughout this project.

This second edition contains essentially all that was included in the ten chapters of the first edition, with a substantial editing of misprints and “misprints.” However, it has grown to have seven more chapters to incorporate some important material that I left out in 1991 and to add new material to reflect some of the new developments. The result is that this is essentially a new book. I hope that in the process of writing this second edition I have not ruined what was good in the first one, and that the new material will be useful to a wide spectrum of people, not only in condensed matter. Although the book is significantly larger than its first edition, I had to leave out some really important material. In particular, I incorporated hardly any discussion of Fermi liquids, non-Fermi liquids (except for Luttinger liquids), and superconductors, among many important problems that are also of interest to me.

Several notable books that cover some parts of the material I cover have appeared in print since 1991, such as Xiao-Gang Wen’s *Quantum Field Theory of Many Body Systems* (published in 2003) and Subir Sachdev’s *Quantum Phase Transitions* (published in 1999). Other books that cover some aspects of the material are Assa Auerbach’s *Interacting Electrons and Magnetism* (published in 1994) and the book by A. Gogolin, A. Nersesyan, and A. Tsvelik, *Bosonization and Strongly Correlated Systems* (published in 2004), as well as the superb *Principles on Condensed Matter Physics* by Paul Chaikin and Tom Lubensky (published in 1995) and John Cardy’s *Scaling and Renormalization in Statistical Physics* (published in 1996).

I am deeply indebted to many people whose work has influenced my views. I have to particularly thank Steve Kivelson for his long-term friendship and collaboration, which has had a strong impact on my work, as reflected here. I also thank my collaborators in many projects, some of which are reflected here, Chetan Nayak, Claudio Chamon, Paul Fendley, Shivaji Sondhi, Joel Moore, and Fidel Schaposnik. I am also indebted to Lenny Susskind and Steve Shenker, who played a great role during my formative years as a theorist and whose outlook has strongly influenced these pages. I also thank my former students Ana López, Christopher Mudry,
Preface to the second edition

Antonio Castro Neto, Eun-Ah Kim, Michael Lawler, Kai Sun, and Benjamin Hsu, whose work is also reflected here. I am also indebted to my colleagues Mike Stone and Rob Leigh, with whom I collaborated in several projects and had countless stimulating discussions. Their work has strongly influenced my own. I also wish to thank Taylor Hughes and Shinsei Ryu for explaining their work (and others) on topological insulators, and motivating me to think on these problems. I am also grateful to Pouyan Ghaemi for reading the chapter on topological insulators and catching several misprints, and to Rodrigo Soto Garrido and to Ponnuraj Krishnakumar for proofreading the entire book and for their great help in generating the skyrmion figures for the cover.

I must also acknowledge the constant and permanent support of the Department of Physics of the University of Illinois, and my colleagues in our department. Some of the material presented here was also used in several special-topics courses I taught in Urbana over the years. I am particularly grateful to Professor Dale van Harlingen, our Department Head, for his constant support. I also wish to thank the many people who over the years have pointed out to me several conceptual issues present in the first edition as well as numerous misprints. I hope the editing of the second edition is substantially better than that of the first. I also wish to thank the National Science Foundation, which supported my research for many years.

This second edition, much like the first, could not have existed without the emotional support and love of Claudia, my wife and lifetime companion. Our children have fortunately (for them) been spared this second edition, which also could not have existed without my father constantly asking when I was going to be done with it.

Eduardo Fradkin
Urbana, Illinois, USA
Preface to the first edition

This volume is an outgrowth of the course “Physics of Strongly Correlated Systems” which I taught at the University of Illinois at Urbana-Champaign during the Fall of 1989. The goal of my course was to present the field-theoretic picture of the most interesting problems in Condensed Matter Physics, in particular those relevant to high-temperature superconductors. The content of the first six chapters is roughly what I covered in that class. The remaining four chapters were developed after January 1, 1990. Thus, that material is largely the culprit for this book being one year late! During 1990 I had to constantly struggle between finalizing the book and doing research that I just could not pass on. The result is that the book is one year late and I was late on every single paper that I thought was important! Thus, I have to agree with the opinion voiced so many times by other people who made the same mistake I did and say, don’t ever write a book! Nevertheless, although the experience had its moments of satisfaction, none was like today’s when I am finally done with it.

This book exists because of the physics I learned from so many people, but it is only a pale reflection of what I learned from them. I must thank my colleague Michael Stone, from whom I have learned so much. I am also indebted to Steven Kivelson, Fidel Schaposnik, and Xiao-Gang Wen, who not only informed me on many of the subjects which are discussed here but, also, more importantly, did not get too angry with me for not writing the papers I still owe them.

This book would not have existed either without the extraordinary help of Christopher Mudry, Carlos Cassanello, and Ana López, who took time off their research to help me with this crazy project. They have done an incredible job in reading the manuscript, finding my many mistakes (not just typos!), making very useful comments, and helping me with the editing of the final version. I am particularly indebted to Christopher, who made very important remarks and comments concerning the presentation of very many subjects discussed here. He also generated the figures. Mrs. Phyllis Shelton-Ball typeset the first six chapters. My wife,
Preface to the first edition

Claudia, made this project possible by learning L\LaTeX{} at great speed and typesetting the last four chapters, correcting some of my very boring and awkward writing style.

This book was also made possible by the love and help of my children Ana, Andrés, and Alejandro, who had to live with a father who became a ghost for a while. Ana and Andrés helped in the proofreading, and took care of their little brother, who helped by keeping everybody happy.

Finally, I must acknowledge the support of the Department of Physics and the Center for Advanced Study of the University of Illinois. The help and understanding of the staff at Addison Wesley is also gratefully acknowledged.

Eduardo Fradkin
Urbana, Illinois, USA