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Introduction

1.1 Field theory and condensed matter physics

Condensed matter physics is a very rich and diverse field. If we are to define it as being "whatever gets published in the condensed matter section of a physics journal," we would conclude that it ranges from problems typical of material science to subjects as fundamental as particle physics and cosmology. Because of its diversity, it is sometimes hard to figure out where the field is going, particularly if you do not work in this field. Unfortunately, this is the case for people who have to make decisions about funding, grants, tenure, and other unpleasant aspects in the life of a physicist. They have a hard time figuring out where to put this subject which is neither applied science nor dealing with the smallest length scales or the highest energies. However, the richness of the field comes precisely from its diversity.

The past few decades have witnessed the development of two areas of condensed matter physics that best illustrate the strengths of this field: critical phenomena and the quantum Hall effect. In both cases, it was the ability to produce extremely pure samples which allowed the discovery and experimental study of the phenomenon. Their physical explanation required the use of new concepts and the development of new theoretical tools, such as the renormalization group, conformal invariance, and fractional statistics.

While the concept of conformal invariance was well known in field theory before critical phenomena became recognized as a field, its importance to the complete structure of the field theory was not understood. The situation changed with the development of the renormalization group (RG). For condensed matter physics, the RG is the main tool for the interpretation of the experimental data, providing the conceptual framework and the computational algorithm which has allowed the theory to make powerful predictions. In particle physics, the RG is also a tool for the interpretation of the data. But, more importantly, the concept of an infrared-unstable fixed point has become the *definition* of the field theory itself.

Similarly, the Chern–Simons theories, which are field theories that describe systems exhibiting fractional statistics, were known before the quantum Hall effect (QHE) was discovered (actually they were discovered at about the same time), but were regarded as a curiosity of field theories below four dimensions: in other words, a beautiful piece of mathematical physics but without relevance to "the world." We have come to recognize that Chern–Simons theories are the natural theoretical framework to describe the quantum Hall effect.

Another case relevant to this point is superconductivity. Viable mechanisms for superconductivity have been known for the fifty-some years that have passed since the theory of Bardeen, Cooper, and Schrieffer (BCS). This theory has successfully explained superconductivity, and a variety of related phenomena, in very diverse areas of physics. This theory has been applied to diverse areas of physics, ranging from superconductivity in metals and superfluidity of liquid ^{3}He in condensed matter physics to neutron stars and nuclear matter in nuclear physics, and dynamical symmetry breaking and grand unification mechanisms (such as technicolor) in elementary-particle physics.

The origin of this constant interplay between field theory and condensed matter (or statistical) physics is that, despite their superficial differences, both fields deal with problems that involve a large (macroscopic) number of degrees of freedom that interact with each other. Thus, it should be no surprise that the *same techniques* can be used in both fields. The traditional trend was that field theory provided the tools (and the "sexy" terms) which were later adapted to a condensed matter problem. In turn, condensed matter models were used as "toy models" in which to try new techniques. Although this is still the case, more recent developments in condensed matter physics have allowed us to investigate new fundamental conceptual problems in quantum field theory. However, as the examples of the RG and the QHE show, the "toy models" can provide a framework for the understanding of much more general phenomenon. The *experimental accessibility* of condensed matter systems is just as important. The MOSFETs and heterostructures in which the QHE is studied have given us the surprisingly exact quantization of the Hall conductance whose understanding has required the use of topology and fiber bundles.

The importance of condensed matter physics to field theory, and vice versa, has been recognized at least since the 1950s. Landau and Feynman are perhaps the two theorists who best understood this deep connection. They worked in both fields and used their ideas and experience from one field in the other and then the converse.

1.2 What has been included in this book (first edition)

This volume is an outgrowth of the course "Physics of Strongly Correlated Systems" which I taught at the University of Illinois at Urbana-Champaign during the

Fall of 1989. Much of the material covered here has been the subject of intense research by a lot of people during the past four years. Most of what I discuss here has never been presented in a book, with the possible exception of some reprint volumes. While the *choice* of the material is motivated by current work on high-temperature superconductors, the methods and ideas have a wide range of applicability.

This book is not a textbook. Many of the problems, ideas, and methods which are discussed here have become essential to our current understanding of condensed matter physics. I have made a considerable effort to make the material largely self-contained. Many powerful methods, which are necessary for the study of condensed matter systems in the strong-fluctuation limit, are discussed and explained in some detail within the context of the applications. Thus, although the theoretical apparatus is not developed systematically and in its full glory, this material may be useful to many graduate students, in order for them to learn both the subject and the methods. For the most part I have refrained from just quoting results without explaining where they come from. So, if a particular method happens to be appropriate to the study of a particular subject, I present a more or less detailed description of the method itself. Thus, various essential theoretical tools are discussed and explained. Unfortunately, I was able to cover only part of the material I wanted to include. Perhaps the biggest omission is a description of conformal field theory. This will have to wait for a second edition, if and when I ever become crazy enough to come back to this nightmare.

The material discussed here includes path-integral methods applied to several problems in condensed matter such as the Hubbard model, quantum spin systems and the fractional quantum Hall effect; $1/N$, $1/S$, and other semi-classical expansions; coherent states; the Bethe ansatz; Jordan–Wigner transformations and bosonization; gauge invariance; topological invariants in antiferromagnets; the Hall effect; and the Chern–Simons theory of fractional statistics. The material is always developed within the context of a particular application. While there is the danger that the application may go "out of fashion," I find that it is easier to motivate and to understand this material within the framework of a concrete problem. Perhaps what this book may be good for is not so much for learning the *techniques* but as a place to find the conceptual framework of field theory in a condensed matter setting.

1.3 What was left out of the first edition

The course that I taught had as its subtitle "High Temperature Superconductors and Quantum Antiferromagnets." As the reader will soon find out, in the material that I have covered there is plenty of quantum antiferromagnetism but little superconductivity. This is not an oversight on my part. Rather, it is a reflection of what we understand today on this subject which is still a wide open field. Thus I chose not to include *the very latest fashion* on the subject but only what appears to be rather well established. This is a field that has produced a large number of very exciting ideas. However, the *gedanken theories* still dominate. To an extent, this book reflects my own efforts in transforming several fascinating *gedanken theories* into something more or less concrete.

Still, the tantalizing properties of the high-temperature superconductors seem to demand from us novel mechanisms such as Phil Anderson's RVB. But, of course, this is far from being universally accepted. After all, with a theory like BCS being around, with so many successes in its bag, it seems strange that anybody would look for any other mechanism to explain the superconductivity of a set of rather complex materials. After all, who would believe that understanding the superconductivity produced by stuff made with copper and oxygen, mixed and cooked just right, would require the development of fundamentally new ideas? Right? Well, maybe yes, maybe not.

1.4 What has been included in the second edition

I have not changed at all the content of what I wrote in the first section of this chapter back in 1991. If anything, these words are even more pertinent today.

Over the years I have often decided that it had been a mistake to include certain topics in the first edition, since they no longer seemed relevant, and regretted not having included others for similar reasons. However, there is a conservation law of good ideas in physics. So it is often the case that a theory that was proposed at a certain time in a certain setting acquires new life and meaning in a different setting. A case in point is the material on spin liquids, both chiral and non-chiral, which was discussed in Chapters 6 and 7 of the first edition. Shortly after the book appeared it became clear that the chiral spin liquid, and the anyon superconductor, do not play a role in the physics of high-temperature superconductors. This was possible since these are examples of internally consistent theories, as opposed to *gedanken* ones, which make clear predictions and hence can be tested in experiment. Nevertheless, the chapters on spin liquids and quantum dimers regained their relevance in the late 1990s and in much of the following decade as evidence for the internal consistency of topological phases in frustrated quantum magnets and quantum dimer models became more established, even though so far they have eluded experimental confirmation.

This second edition is in many ways a new book. Here is a summary of what has been included. Except for correcting a few misprints and typos, Chapter 2, The Hubbard model, is the same as in the first edition. Chapter 3, The magnetic instability of the Fermi system, has been edited to remove typos and misprints and the last section has had its mistakes purged. Chapter 4, The renormalization group, is new. In the first edition the discussion on the renormalization group was scattered throughout the text. In Chapter 4 I present a succinct but modern presentation of the subject, which sets the stage for its use in other chapters. This chapter was strongly influenced by John Cardy's beautiful textbook (Cardy, 1996). Chapter 5, One-dimensional quantum antiferromagnets, was edited and revamped. It now has three sections discussing the important subject of duality in spin systems, and another one on the one-dimensional quantum Ising model, including the exact solution. The section on Abelian bosonization was updated, particularly the notation. Chapter 6, The Luttinger liquid, is entirely new. Although some of this material also appears in Chapter 5, here I give what I think is a thorough presentation of this important problem in condensed matter. Some of the material used here is strongly inspired by reviews written by Kivelson and Emery on this problem (Emery, 1979; Carlson *et al.*, 2004). Chapter 7, Sigma models and topological terms, is a vastly revised version of what was Chapter 5 in the first edition. The main changes in this chapter are the new sections on the Wess–Zumino–Witten model and non-abelian bosonization, and another section giving a brief presentation of the main ideas of conformal field theory (a subject that has acquired widespread use in many areas of condensed matter) and their application to the Wess–Zumino–Witten model and to quantum spin chains. For the sake of brevity I chose not to include a discussion of the Kondo problem here.

In the first edition Chapters 6 and 7 dealt with spin liquids and chiral spin states, respectively. These two chapters have been completely revised, expanded and split into three chapters, Chapters 8, 9, and 10. Chapter 8, Spin-liquid states, contains much of the discussion of the old Chapter 6 on spin liquids, valence-bond states, and the gauge-theory description of antiferromagnets, but significantly edited and updated to account for the many developments. The content of the new Chapter 9, Gauge theory, dimer models, and topological phases, is completely new. Here I include an in-depth discussion of the phases and observables of gauge theories, paying special attention to their relation to time-reversal-invariant topological phases, the $\mathbb{Z}_2$ spin liquid, the Kitaev toric code, and quantum loop models. I also include a theory of quantum criticality in quantum dimer models and the quantum Lifshitz model.

In the new Chapter 10, Chiral spin states and anyons, I have merged all the discussions on the chiral spin liquid. I also expand the treatment of the Chern–Simons gauge theory and its role as a theory of fractional statistics. I also corrected some errors on the lattice version of Chern–Simons that were present in the first edition. Chapter 11, Anyon superconductivity, is a compressed version of Chapter 8 of the first edition. It is now clear that an anyon superconductor, a state resulting from the condensation of electrically charged anyons (abelian) is not essentially different from a superconductor with a spontaneously broken time-reversal symmetry, e.g. a

$p_x + ip_y$ or $d_{x^2-y^2} + id_{xy}$ superconductor. Nevertheless, I am not fond of rewriting history and for this reason I kept this chapter, after excising some results that were wrong.

Chapter 12, Topology and the quantum Hall effect, is almost the same as Chapter 9 of the first edition. I only made minor editing changes. Similarly, the new Chapter 13, The fractional quantum Hall effect, is almost the same as Chapter 10 in the first edition. The only important change here, aside from editing, was that the section on edge states is no longer in this chapter. The bulk of this chapter is devoted to a presentation of the bosonic and fermionic Chern–Simons theory of the fractional quantum Hall states.

The remaining four chapters of the new edition are new and are devoted to, respectively, Topological fluids (Chapter 14), Physics at the edge (Chapter 15), Topological insulators (Chapter 16), and Quantum entanglement (Chapter 17). Chapter 14 is devoted to the theory of topological fluids presented here as a theory of fractional quantum Hall fluids. Here I include a description of the hydrodynamic theory (of Wen and Zee), its extensions to general abelian multi-component fluids, non-abelian quantum Hall fluids, superconductors as topological fluids, and topological superconductors, and a brief presentation of the concepts of braiding and fusion. Chapter 15, Physics at the edge, is an in-depth presentation of the theory of edge states in integer and fractional quantum Hall fluids, both abelian and non-abelian, Wen's theory of bulk–edge correspondence, and the effective field theories of the non-abelian fractional quantum Hall states. I devote special sections to discussions of tunneling conductance at quantum point contacts, noise and the measurement of fractional charge, and the theory of abelian and non-abelian quantum interferometers, and there is a brief sales pitch for topological quantum computing. Chapter 16 is devoted to a brief presentation of the exciting new field of topological insulators. Here I discuss the basic concepts, band topological invariants, the anomalous quantum Hall effect, and the spin quantum Hall effect and its experimental discovery. I also discuss the extensions of these ideas to three-dimensional $\mathbb{Z}_2$ topological insulators, their relations to fractional charge (and polyacetylene) in one dimension, the Callan–Harvey effect in three dimensions, surface Weyl fermions, Majorana modes, and possible new topological insulators resulting from spontaneous symmetry breaking. Chapter 17 is devoted to the role of quantum entanglement in field theory, quantum critical systems and topological phases, and large-scale entanglement and the scaling of the entanglement entropy, as well as the relation of this problem to the modern ideas of holography and the CFT/gravity duality.

Several important subjects are not in this book. In particular, except for some cursory discussion in Chapter 2, Fermi liquids are not discussed. For this reason I have also not discussed the Bardeen–Cooper–Schrieffer theory of superconductivity and other mechanisms. I have also not discussed what happens when a Fermi liquid fails. This is an area to which I have devoted a great deal of effort, including the formulation of a new class of electronic liquid-crystal phases of strongly correlated systems. Experimentally these phases arise often in conjunction or in competition with high-temperature superconductivity. I have also not discussed the important problem of fermionic quantum criticality and non-Fermi-liquid behavior, with many interesting connections with the concept of holography, as well as extensions of bosonization to dimensionalities higher than two, which is a natural framework to describe these open problems. Absent from this book is also the discussion of disordered systems, a fascinating problem in which there are very few well-established results.

Finally, all the figures in this book are new because I had lost the source files of the figures that were used in the first edition. This new edition includes an extensive set of references at the end of the book, and a detailed index, which I hope will be useful to the reader.

2

The Hubbard model

2.1 Introduction

All theories of strongly correlated electron systems begin with the Hubbard model because of its simplicity. This is a model in which *band electrons* interact via a two-body *repulsive* Coulomb interaction. No phonons are present, and in general no explicitly attractive interactions are included. For this reason, the Hubbard model has traditionally been associated with magnetism. Superconductivity, on the other hand, has traditionally (i.e. after BCS) been interpreted as an instability of the ground state resulting from *effectively attractive* interactions (say, electron–phonon as in BCS). A novel situation has arisen with Anderson's suggestion (Anderson, 1987) that the superconductivity of the new high-T_c materials may arise from purely repulsive interactions. This suggestion was motivated by the fact that the superconductivity seems to originate from doping (i.e. extracting or adding charges) an otherwise insulating state.

The Hubbard model is a very simple model in which one imagines that, out of the many different bands which may exist in a solid, only very few states per unit cell contribute significantly to the ground-state properties. Thus, if a Bloch state of energy ϵ_p, momentum $\vec{p}$, and index α has a wavefunction $\Psi_{\vec{p},\alpha}$, one can construct Wannier states

$$\Psi_\alpha(\vec{r}_i) = \frac{1}{\sqrt{N}} \sum_{\vec{p}\in \mathrm{BZ}} e^{i\vec{p}\cdot\vec{r}_i} \Psi_{\vec{p},\alpha}(\vec{r}_i) \tag{2.1}$$

where $\vec{r}_i$ is the location of the ith atom and BZ is the Brillouin zone. The assumption here will be that only one (or a few) band indices matter, so I will drop the index α. The Coulomb interaction matrix elements are

$$U_{ij,i'j'} = \int d^3r_1\, d^3r_2\, \Psi_i^*(\vec{r}_1)\Psi_j^*(\vec{r}_2)\tilde{V}(\vec{r}_1 - \vec{r}_2)\Psi_{i'}(\vec{r}_1)\Psi_{j'}(\vec{r}_2) \tag{2.2}$$

(in three dimensions), where $\tilde{V}$ is the (screened) Coulomb interaction. Since $\tilde{V}$ is expected to decay as the separation increases, the largest term will be the "on-site" term: $U_{ii,ii} \equiv U$. Next will come nearest neighbors, etc. Moreover, since the Wannier functions have exponentially decreasing overlaps, $U_{ij,i'j'}$ is expected to decrease rather rapidly with the separation $|i-j|$.

The second quantized Hamiltonian tight binding (in the Wannier-functions basis) is

$$\begin{aligned} H = &- \sum_{\substack{\vec{r}_i,\vec{r}_j \\ \sigma=\uparrow,\downarrow}} \left(c^\dagger_\sigma(\vec{r}_i) t_{ij} c_\sigma(\vec{r}_j) + c^\dagger_\sigma(\vec{r}_j) t_{ij} c_\sigma(\vec{r}_i) \right) \\ &+ \frac{1}{2} \sum_{\substack{i,j,i',j' \\ \sigma,\sigma'=\uparrow,\downarrow}} U_{ij,i'j'} c^\dagger_\sigma(\vec{r}_i) c^\dagger_{\sigma'}(\vec{r}_j) c_{\sigma'}(\vec{r}_{j'}) c_\sigma(\vec{r}_{i'}) \end{aligned} \tag{2.3}$$

where $c^\dagger_\sigma(\vec{r})$ creates an electron at site $\vec{r}$ with spin σ (or more precisely, at the unit cell $\vec{r}$ in the band responsible for the Fermi surface) and satisfies

$$\begin{aligned} \{c_\sigma(\vec{r}), c^\dagger_{\sigma'}(\vec{r}\,')\} &= \delta_{\sigma,\sigma'}\delta_{\vec{r},\vec{r}\,'} \\ \{c_\sigma(\vec{r}), c_{\sigma'}(\vec{r}\,')\} &= 0 \end{aligned} \tag{2.4}$$

The Hubbard model is an approximation to the more general Hamiltonian, Eq. (2.3), in which the hopping is restricted to nearest neighboring sites:

$$t_{ij} = \begin{cases} t & \text{if } i,j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

and the Coulomb interaction is assumed to be screened. If just the "on-site" term is kept,

$$U_{ij,i'j'} = U\delta_{ij}\delta_{i'j'}\delta_{ii'} \tag{2.6}$$

the resulting model Hamiltonian

$$H = -t \sum_{\substack{\langle\vec{r},\vec{r}\,'\rangle \\ \sigma=\uparrow,\downarrow}} \left(c^\dagger_\sigma(\vec{r}) c_\sigma(\vec{r}\,') + \text{h.c.} \right) + U \sum_{\vec{r}} n_\uparrow(\vec{r}) n_\downarrow(\vec{r}) \tag{2.7}$$

is known as the one-band Hubbard model. In Eq. (2.7), we have dropped the lattice site labels and $\langle\, , \rangle$ means nearest-neighboring sites. This is the tight-binding approximation and represents the one-band Hubbard model. We have introduced

$$n_\sigma(\vec{r}) = c^\dagger_\sigma(\vec{r}) c_\sigma(\vec{r}) \tag{2.8}$$

From the Pauli principle we get $n_\sigma = 0, 1$ or $n_\sigma^2 = n_\sigma$ at every site.

The Hilbert space of this system is the tensor product of only *four* states per site, representing $|0\rangle$ as nothing, $|\uparrow\rangle$ as an electron with spin up, $|\downarrow\rangle$ as an electron with

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Eduardo Fradkin
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spin down, and $|\uparrow\downarrow\rangle$ as an up–down pair. The states $|0\rangle$ and $|\uparrow\downarrow\rangle$ are spin singlets (i.e. $S = 0$).

It is convenient to define the following operators. The spin operator $\vec{S}(\vec{r})$ is defined by (the summation convention is assumed)

$$\vec{S}\,(\vec{r}) = \frac{\hbar}{2} c^\dagger_\sigma(\vec{r}) \vec{\tau}_{\sigma\sigma'} c_{\sigma'}(\vec{r}) \tag{2.9}$$

where $\vec{\tau}$ are the (three) Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.10}$$

The particle number operator at site $\vec{r}$ (or *charge*) is

$$n(\vec{r}) = \sum_\sigma n_\sigma(\vec{r}) = \sum_\sigma c^\dagger_\sigma(\vec{r}) c_\sigma(\vec{r}) \equiv c^\dagger_\sigma(\vec{r}) 1_{\sigma\sigma'} c_{\sigma'}(\vec{r}) \tag{2.11}$$

and the associated total charge Q is given by

$$Q = e \sum_{\vec{r}} n(\vec{r}) \equiv e N_{\rm e} \tag{2.12}$$

2.2 Symmetries of the Hubbard model

2.2.1 SU(2) spin

Suppose we rotate the local spin basis (i.e. the quantization axis)

$$c'_\sigma(\vec{r}) = U_{\sigma\sigma'} c_{\sigma'}(\vec{r}) \tag{2.13}$$

where U is a 2×2 SU(2) matrix. Namely, given four complex numbers a, b, c, and d, the matrix U given by

$$U = \begin{pmatrix} a & c \\ d & b \end{pmatrix} \tag{2.14}$$

must satisfy

$$U^{-1} = U^\dagger \equiv \left(U^{\rm T}\right)^* \tag{2.15}$$

together with the condition

$$\det U = 1 \tag{2.16}$$

We will parametrize the matrix $U(\vec{\theta})$ as follows:

$$U(\vec{\theta}) = e^{i\vec{\theta}\cdot\vec{\tau}} = \mathbf{1} \cos|\vec{\theta}| + i \sin|\vec{\theta}| \frac{\vec{\theta}\cdot\vec{\tau}}{|\vec{\theta}|} \tag{2.17}$$