1

Flexagons – A beginning thread

1.1 Four scientists at play

In 1939 four young men were thrown together in graduate school at Princeton. They came from diverse backgrounds and went on to have very different careers. But for a short while all of them played with straight strips of paper made into what became known as "flexagons." The thread that runs through this story is that of four creative young men who played for a while with an intriguing toy and then went on to other creative ventures in mathematics, physics, statistics, and computer science. We present here a small bit from each of their life stories.

In 1939 Arthur H. Stone (1916–2000), then a newly transplanted Englishman, was beginning his PhD work with Solomon Lefschetz at Princeton. According to Paul M. Cohn's obituary of Stone [6],

> Arthur Harold Stone . . . was one of the foremost general topologists of his time, and made significant contributions to a number of different parts of general topology. . . . In 1927 [he] won a LCC scholarship to Christ's Hospital (Horsham). This was a boarding school which has had such successful pupils as Philip Hall, Christopher Zeeman (later Sir Christopher) and D. G. Northcott (Stone's contemporary). The mathematics teaching was in the hands of C. A. J. Trimble, himself a Wrangler. Here, Arthur won prizes in almost all subjects except sports (though he was also good at rugger [rugby football]).
>
> In 1935 he gained a major scholarship to Trinity College, Cambridge. He excelled at the academic subjects, but was also an outstanding violinist and good at chess. At Cambridge he continued with the violin and became leader of the orchestra of the Cambridge University Music Society. He was a Wrangler, and took his BA in 1938 before going to Princeton to work for a PhD with S. Lefschetz.
>
> To fit the American notebook sheets into his English binder, he had to trim off an inch of paper, and he began to fold these strips in various ways. This led to some intriguing figures, which later became famous as 'flexagons' (see [17]). He was both very inventive and also adept with his hands, talents which he used in building a counterclockwise grandfather clock.

The popularization of flexagons began when Martin Gardner wrote about them in his first ***Scientific American*** article [16]; and it was that article that led to his regular ***Scientific American*** feature article on Mathematical Games. Gardner's article was closely followed by Oakley and Wisner's more mathematical paper on flexagons [54].

The quote in the first obituary above, from [17], gives a fascinating account of Stone's discovery and his collaboration with Bryant Tuckerman (1915–2002), Richard P. Feynman (1919–1988), and John W. Tukey (1915–2000). Each of these collaborators became a famous scientist in his own right – but none of their subsequent successes seems to have been directly concerned with flexagons.

Bryant Tuckerman's obituary [75] states that

> His graduate studies were interrupted by World War II, during which he worked at the U. S. Office of Scientific Research and [the] Office of Scientific Research and Development on navigational devices for tanks. After the war he completed his PhD at Princeton in topology. . . . He then worked for five years at the Institute for Advanced Study in Princeton with John von Neumann on applications of such early computers as the MANIAC. . . . In 1962 Tuckerman published ***Planetary, Lunar, and Solar Positions***, a set of tables covering the years from 601 B.C. to 1649, which is still used by historians and archaeologists to date ancient documents containing astronomical references, from Babylonian times through the Renaissance. In 1971 he found the 24th Mersenne prime, $2^{199937} - 1$, then the largest known prime number.

Richard Feynman was awarded the Nobel Prize in physics in 1965, along with Shinichero Tomonaga of Japan and Julian Schwinger of Harvard University. On his death it was reported by Chandler in Feynman's obituary [5] that

> He was widely known for his insatiable curiosity, gentle wit, brilliant mind and playful temperament. These qualities were clearly evident in his popular 1985 book, "Surely You're Joking Mr. Feynman," which was on the New York Times best-seller list for 14 weeks. . . . Ever playful and unintimidated by authority, Mr. Feynman caused consternation in his years with the Manhattan Project, which developed the atomic bomb, by figuring out in his spare time how to pick the locks on filing cabinets that contained classified information. Without removing anything, he left taunting notes to let officials know that their security system had been breached. . . .
>
> Mr. Feynman attracted widespread attention during the Rogers Commission hearings on the Challenger space shuttle accident in 1986. Frustrated by witnesses' vague answers and by slow bureaucratic procedures, he conducted an impromptu experiment that proved a key point in the investigation: He dunked a piece of the rocket booster's O-ring material into a cup of icewater and quickly showed that it lost all resiliency at low temperatures. . . .
>
> MIT physicist Philip Morrison called Mr. Feynman "the most original theoretical physicist of our time," according to a report by United Press International. Morrison said Mr. Feynman, who called his Nobel Prize "a pain in the neck" was "extraordinarily honest with himself and everyone else," and added that "he didn't like ceremony or pomposity . . . he was extremely informal. He liked colorful language and jokes."

As for John Tukey, ***The New York Times*** reported in 2000:

In 1936, Mr. Tukey graduated from nearby Brown University with a bachelor's degree in chemistry, and in the next three years earned three graduate degrees, one in chemistry at Brown, and two in mathematics at Princeton, where he would spend the rest of his career. At the age of 35 he became a full professor, and in 1965 he became the founding chairman of Princeton's statistics department.

[He was known as] one of the most influential statisticians of the last 50 years and a wide-ranging thinker. . . . Three decades before the founding of Microsoft, Mr. Tukey saw that "software" as he called it, was gaining prominence. "Today", he wrote at the time it is "at least as important" as the "hardware" of tubes, transistors, wires, tapes and the like. . . . Twelve years earlier, while working at Bell Laboratories he had coined the term "bit," an abbreviation of "binary digit" that described the 1's and 0's that are the basis of computer programs. Both words caught on, to the chagrin of some computer scientists who saw Mr. Tukey as an outsider. "Not everyone was happy that he was naming things in their field," said Steven M. Schultz, a spokesman for Princeton. . . .

As his career progressed, he also became a hub for other scientists. He was part of a group of Princeton professors that gathered regularly and included Lyman Spitzer, Jr., who inspired the Hubble Space Telescope.

His first brush with publicity came in 1950, when the National Research Council appointed him to a committee to evaluate the Kinsey Report, which shocked many Americans by describing the country's sexual habits as far more diverse than had been thought. From their first meeting, when Mr. Kinsey told Mr. Tukey to stop singing a Gilbert and Sullivan tune aloud while working, the two men clashed, according to "Alfred C. Kinsey", a biography by James H. Jones.

In a series of meetings over two years Mr. Kinsey vigorously defended his work which Mr. Tukey believed was seriously flawed, relying on a sample of people who knew each other. Mr. Tukey said a random selection of three people would have been better than a group of 300 chosen by Mr. Kinsey.

Nothing that any of these four brilliant men did would have led one to suppose that they would jointly invent a very special brand of polygon, namely, a flexagon. It is interesting to point out that, of the four pioneers of flexagons, Arthur Stone was the only professional mathematician.

1.2 What are flexagons?

In the next section we will describe in detail how to construct flexagons; that is, special polygons that change their appearances when they are manipulated in certain ways. But first we need some terminology.

In general, we refer to these configurations as N-*flexagons*, where N indicates the number of congruent triangles surrounding the center of the regular polygon formed by the completed construction. We should point out that in the case of the 8-flexagon, the bounding polygon of the construction has 4 sides, not 8 as you might expect. For this reason the 8-flexagon is often referred to in the literature as a *tetraflexagon*.

We first describe in detail how to construct and flex two special, different types of flexagon, for each of the values $N = 6$ and $N = 8$. We also refine the nomenclature by adding another number at the beginning of the name to indicate how many different complete "faces," of N triangles each, the model may present when it is flexed. This nomenclature will be illustrated and described in detail at the appropriate time – and, for those of you who like to use big words, we will also give you the names that utilize Greek prefixes instead of numbers.

We first discuss, in Section 1.3, the hexaflexagons (which we call ***6-flexagons***). We do this because they are the easiest flexagons to manipulate. Our idea is that you are likely to find it more pleasant to develop your flexing skills on the 6-flexagons, and that by doing this you will be better prepared to appreciate the considerably more complicated octaflexagons (or 8-flexagons) that follow in Section 1.4. Of course, in keeping with the spirit of this entire book, we suggest variations or references along the way so that you can construct (or maybe even *invent*) other flexagons on your own.

1.3 Hexaflexagons

Required materials

- Strips (or a roll) of gummed mailing tape or adding machine tape about $1\frac{1}{2}$ inches (4 cm) wide. The glue on the gummed tape should be of the type that needs to be moistened to become sticky.
- White glue, or a glue stick, if your folding tape is not gummed.

The simplest hexaflexagon is made from a straight strip of 10 equilateral triangles (if you want an easy way to construct these triangles see Section 2.3). We describe the construction here in terms of instructions of a kind that we will use to show how to make any of our flexagons.

Basic instructions

1. Prepare the pattern piece, labeling both sides of it *precisely* as shown.
2. Crease all fold lines in *both* directions.
3. Beginning with the strip as shown at the top of Figure 1.1, fold *in order* (so that the numbers are no longer visible),

   triangle 1 onto triangle 1,
   triangle 2 onto triangle 2,
   triangle 3 onto triangle 3,
   •
   •
   •
   and, finally, triangle ☆ onto triangle ☆.
4. Glue, or attach with a paper clip, so that ☆ is attached to ☆.
5. Gently flex and play with your model – decorate the faces with interesting patterns.

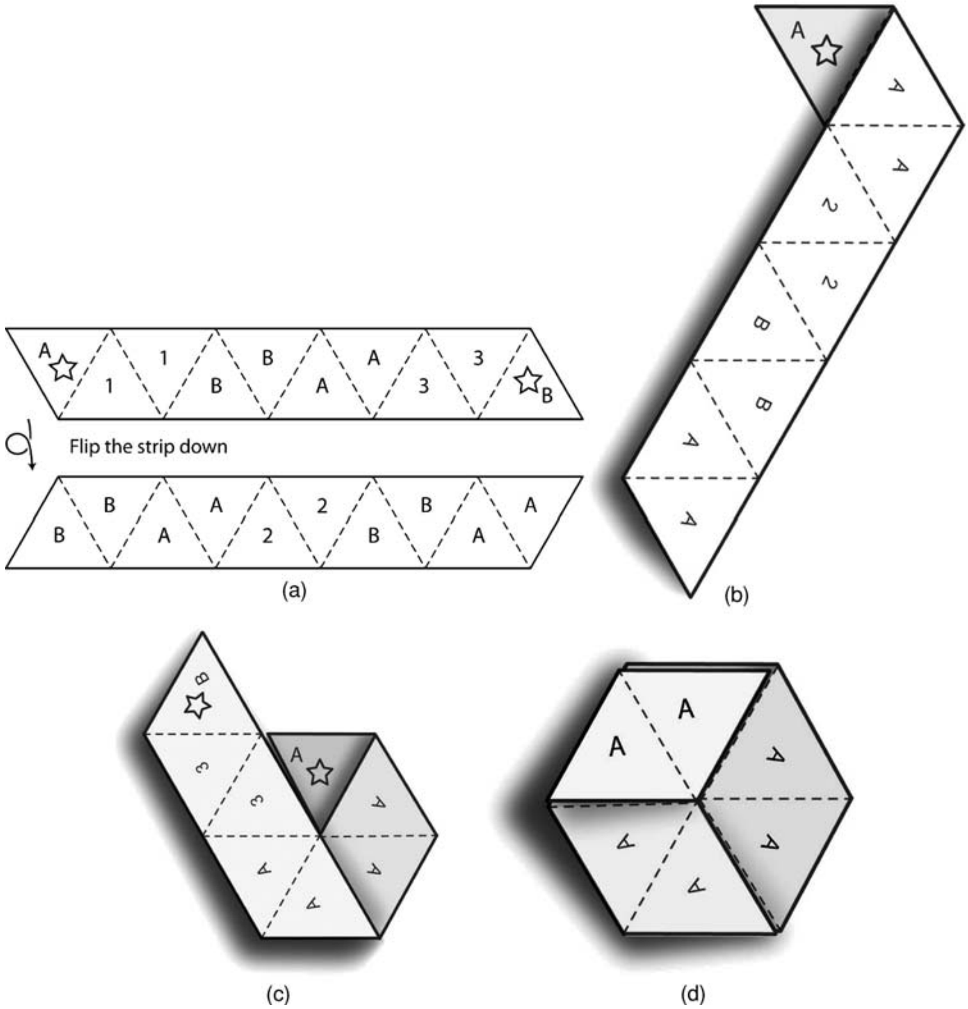

Figure 1.1 (a) Labeling the pattern piece for the 6-flexagon. (b) The construction with triangle 1 on triangle 1, (c) with triangle 2 on triangle 2, (d) with triangles 3 on triangle 3 and nearly completed.

Figure 1.1(a) shows the pattern piece for the 6-flexagon. A word of caution: Note that when you “flip the strip *down*,” as indicated by the symbol between the two strips, you simply take the top edge and lift it off the table and place it at the bottom (nothing moves in the right- or left-hand direction). Place the strip on the table with the side showing the numbers 1 and 3 visible. Then follow the instructions. Figures 1.1(b)–(d) should reassure you that you are doing it correctly. Try out the construction.

Now for the magic! Gently mountain-fold and valley-fold the hexagon as shown in Figure 1.2(a) to make a 3-petaled arrangement that will “come apart” at the

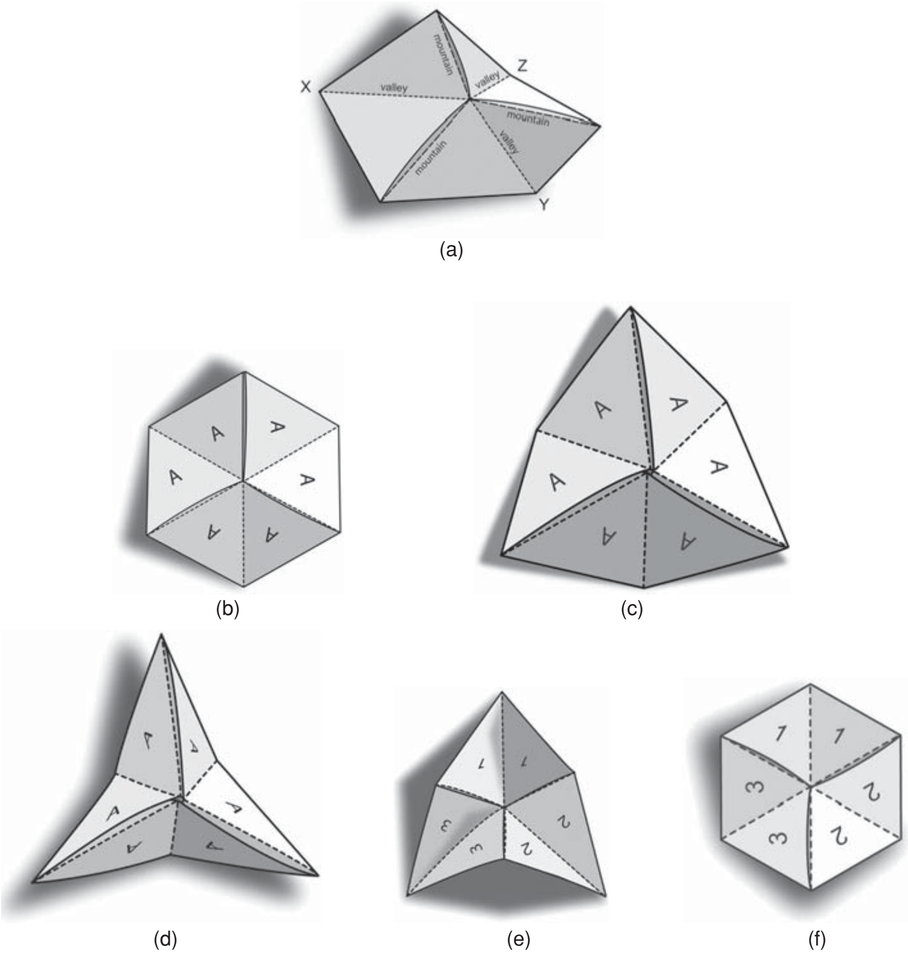

Figure 1.2 (a) Beginning to flex. Note the location of the "slits" on the mountain folds. (b)–(f) How the flex proceeds, showing the labeled faces.

top and lie flat when the vertices labeled x, y, z are brought together *below* the hexagon. Following the illustrations in Figures 1.2(b)–(f) may be helpful. Repeat the process. Notice that as you flex the hexagon in this way you eventually see 3 faces, the A face, the B face, and the 1-2-3 face.

Although 6-flexagons constructed from the same width tape will all have the same shape and size, they may differ in the number of hexagonal faces that can be presented as the polygon is flexed. In this sense, the flexagon you have just constructed is the "smallest" hexaflexagon that can be constructed with a *straight* strip of equilateral triangles. Since it has 3 faces, we call it a 3-6-flexagon (it is well-known by the name of ***trihexaflexagon***).

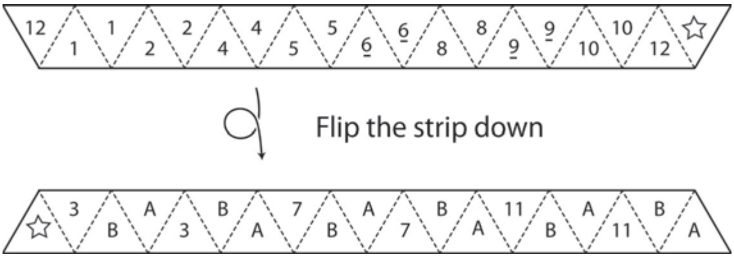

Figure 1.3 Pattern piece for the 6-6-flexagon.

Play with your flexagon until you become adept at flexing it. You may want to draw some patterns on its faces. Begin by drawing a design on the two visible faces and then flex it. You will notice a blank face appears and one of the existing faces disappears – and even the face that is still visible may seem changed. You will soon see that although you can draw patterns on only 3 hexagonal faces originally, more than 3 designs will appear, owing to the way the patterns on the triangular portions of the face are moved about when the flexagon is flexed.

A 6-6-flexagon (the ***hexahexaflexagon***) may be constructed from a strip of 19 equilateral triangles. The pattern piece is shown in Figure 1.3. Now it's up to you! Following the basic instructions, make your 6-6-flexagon and then read the rest of this section for suggestions about how to flex it and how to build even bigger 9-6-flexagons, 12-6-flexagons, and, in general, $3n$-6-flexagons. ***A practical hint***: As you complete each "triangle" so that an A is on one side and a B is on the other side, place a paper clip on that triangle and continue to fold the next triangle, placing like numbers on each other. When you have completed the flexagon remove all the paper clips. At that point there should be 6 triangles labeled A on one face and 6 triangles labeled B on the other face.

The 6-6-flexagon is flexed in exactly the same way as the 3-6-flexagon. However, most people have difficulty finding all 6 faces. Bryant Tuckerman invented a procedure for bringing out the 6 faces with the shortest possible flexing sequence. His process, known as the ***Tuckerman traverse***, involves continually flexing at one vertex until the flexagon refuses to open, then moving to an adjacent vertex (either way) and continuing to flex at that vertex until the flexagon again refuses to open, and so on. It is an interesting exercise to record the *sequence of faces* that appears as you perform this flexing algorithm. Try it and compare your results with the diagram in Figure 1.4.

Notice that although you drew 6 patterns on the faces of this polygon, there are many more actual designs (since the triangular parts of the hexagon appear in different orientations as you flex the model). How many different designs do you

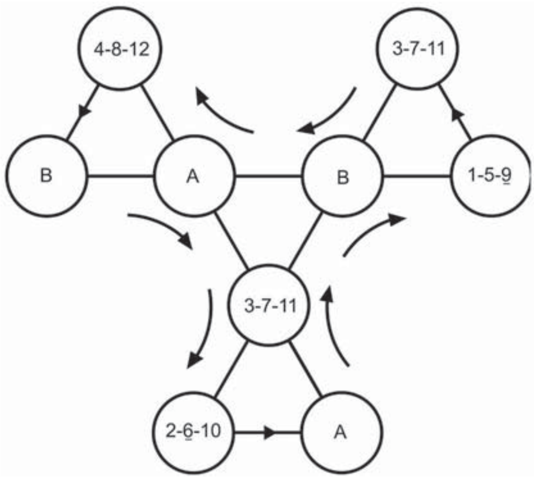

Figure 1.4 The order in which the faces of the 6-6-flexagon appear.

get from your 6 faces? (Your answer will depend on the symmetry of the patterns you use.)

You may have already noticed a pattern in the *number of faces* on these 6-flexagons. The smaller had 3 faces, and the last one had 6 faces. It is a fact (which we don't prove) that the number of faces that can occur on a 6-flexagon constructed with straight strips of equilateral triangles must be a multiple of 3. So, of course, the next larger 6-flexagon will have 9 faces. But, how do we construct it? Here are some hints.

First of all let us suppose that somehow we remember that it is possible to construct a 3-6-flexagon but that we've forgotten how many triangles we need. We can readily calculate the number of triangles required. We need to have 6×3 triangles available in order to provide 3 faces. We also need 2 extra triangles that get glued together. Thus this flexagon contains $6 \times 3 + 2$ triangles in all. However, since each triangle on the strip of tape has 2 sides, the number of triangles this model actually requires is only half this number, that is

$$\frac{\mathbf{3} \times 6 + 2}{2} = 3 \times 3 + 1 = 10.$$

In exactly the same way, we can reason that for a 6-faced 6-flexagon the number of triangles required is

$$\frac{\mathbf{6} \times 6 + 2}{2} = 3 \times 6 + 1 = 19,$$

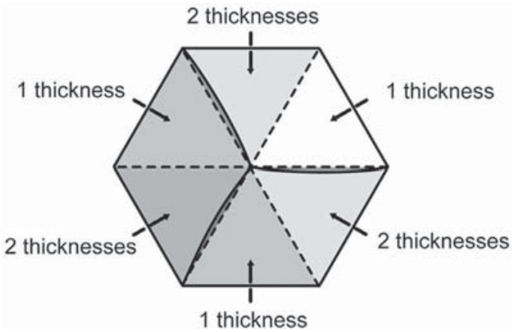

Figure 1.5 $1 + 2 = 3 =$ the number of faces for a 3-6-flexagon.

so that, for the $\mathbf{3n}$-faced 6-flexagon the number of triangles required is

$$\frac{\mathbf{3n} \times 6 + 2}{2} = 3n \times 3 + 1 = 9n + 1.$$

Thus, for example, the 9-6-flexagon (*nonahexaflexagon*) requires a strip of $9 \times 3 + 1 = 28$ triangles.

Now that we know the number of triangles required for our 9-6-flexagon, how do we get them folded in the right arrangement? Again, we study the two 6-flexagons we've already constructed.

Notice that, in the flattened position of the 3-6-flexagon, the thicknesses of tape on two adjacent triangular sections are 1 and 2, respectively (see Figure 1.5). (Where two triangles are glued together, they behave as 1 thickness of tape.)

What is the situation with the 6-6-flexagon? We observe that in its flattened position immediately after construction (*before* any flexing takes place), the thicknesses of tape on two adjacent triangular sections are 2 and 4. However, when the 6-6-flexagon is flexed, it sometimes has thicknesses of 1 and 5 on adjacent triangular sections. See Figure 1.6.

This information contains the secret for constructing the 9-6-flexagon. What we might seek is an arrangement so that the thicknesses on any two adjacent triangular sections of the flattened hexagon sum to 9. One possibility is to use the fact that $4 + 5 = 9$ and try to find out how to fold the strip of equilateral triangles so as to produce adjacent triangles on the finished model having 4 and 5 thicknesses, respectively (see Figure 1.7). But we already know, from our construction of the 6-6-flexagon, how to fold the strip to obtain 4 thicknesses on one of the triangular sections; and, as we've observed, there must exist a way to obtain 5 thicknesses on a triangular section. The idea is to construct the 6-6-flexagon, except that you attach the two faces together with a paper clip instead of using glue. You can then flex this flexagon until you have thicknesses of 1 and 5 on adjacent triangular sections.

Cambridge University Press
978-0-521-76410-0 - A Mathematical Tapestry: Demonstrating the Beautiful Unity of Mathematics
Peter Hilton and Jean Pedersen
Excerpt
More information

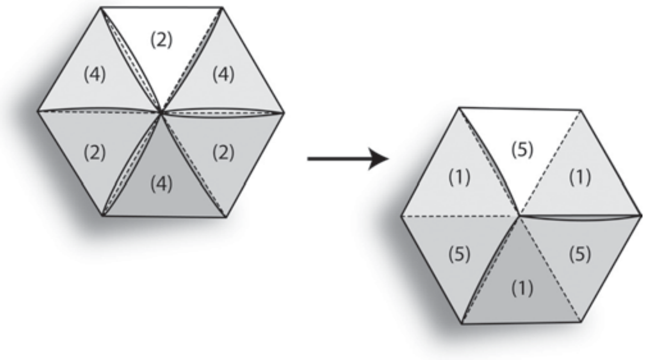

Figure 1.6 $2 + 4 = 1 + 5 = 6 =$ the number of faces for the 6-6-flexagon.

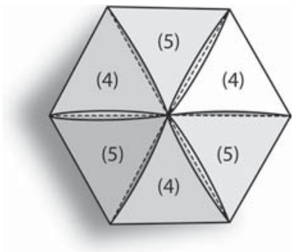

Figure 1.7 Thicknesses for the triangular sections of the 9-6-flexagon.

At that point you can remove the paper clip and “unwrap” the arrangement to *see* how to fold a triangular section with 5 thicknesses. With a little practice you will be surprised how easily you can guess how to fold the required number of thicknesses for a given triangular section.

In the same way that we figure out from the 6-6-flexagon how to construct the 9-6-flexagon with a strip of 28 equilateral triangles, we can use the 9-6-flexagon to discover how to build the 12-6-flexagon (*dodecahexaflexagon*) with a strip of 37 equilateral triangles. Of course, the process goes on, and you may even find that if you try it you can construct the 15-6-flexagon with a strip of 46 equilateral triangles. In fact, constructing it may be easier for some people than learning how to pronounce its Greek name, which is *pentacaidecahexaflexagon*! (*Pentacaideca* means “5 and 10” or “15” and, of course, “hexa” means “6”.)

On a practical note we should warn you that if you construct flexagons with more than 6 faces it is best to trim a small amount of paper from each edge of the strip after folding the triangles before constructing the flexagon, so that the thickness of the paper doesn't get in the way of either the construction or the flexing of the finished product. We can't specify how much you should trim, because the