Cambridge University Press
978-0-521-76360-8 - Nonlinear Resonance Analysis: Theory, Computation, Applications
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Exposition

"But there's need for some proof..." Berlioz began. "There's no need for any proofs," replied the professor and he began to speak softly, while his accent for some reason disappeared: "It's all very simple...

M. Bulgakov
Master and Margarita

1.1 An easy start

As with every new theory, nonlinear resonance analysis has not come *ex caelo*. On the contrary, as Newton said: "If I have seen further it is only by standing on the shoulders of giants." The giants to be grateful to now are Galileo Galilei (1564–1642), Jean Baptiste Joseph Fourier (1768–1830), and Jules Henri Poincaré (1854–1912). The father of modern physics, an Italian, Galileo Galilei, fascinated by the movement of a simple pendulum, identified one of the most important natural phenomena – resonance. French politician and mathematician Fourier, trying to understand what happens when a hot piece of metal rod is put into water, developed the mathematical apparatus – Fourier analysis – for describing solutions to linear partial differential equations (PDEs), without which no area of contemporary science is conceivable. Another French mathematician, Henri Poincaré, used Fourier analysis in order to give a strict mathematical definition of resonance and developed a method – Poincaré transformation – allowing, under some assumptions, to reduce the search for solutions to a nonlinear PDE to the search for resonances. This way the foundations of nonlinear resonance analysis were laid.

Below we begin with the mathematical problem and will return to the pendulum in Chapter 5, for this is a very handy object to illustrate even the quite complicated mathematical results presented in this book.

Fourier analysis

Thus, in the beginning was Fourier analysis. More precisely, in the beginning was a discussion. From 1750–1760, d'Alembert, Euler, Bernoulli, Clairaut, and Lagrange were involved in a prolonged and heated controversy about solutions to the equation of a vibrating string:

$$\psi_{tt} - \alpha^2 \psi_{xx} = 0 \tag{1.1}$$

with some constant α. D'Lambert and Euler derived a functional form of the solution

$$\psi = \varphi_1(x + \alpha t) + \varphi_2(x - \alpha t) \tag{1.2}$$

(on an infinite line), while Bernoulli was the first to present the solution in the form of a trigonometric series of sines and cosines of multiple variables:

$$\psi = A_1 \sin x \cos \alpha t + A_2 \sin 2x \cos 2\alpha t + \cdots \tag{1.3}$$

Bernoulli stated that his solution (1.3) included solution (1.2) as a particular case. Euler disagreed testily, his arguments being that if this were true, an arbitrary function of one variable could be presented as a series of sines, which obviously is not possible because an arbitrary function is not necessarily odd and periodic. With the discussion at this stage, the young and still unknown mathematician Lagrange appeared on the scene and tried to prove that the solution in the functional form (1.2) is more general than Bernoulli's trigonometric series (1.3). By an irony of fate, Lagrange did not notice that at some intermediate step in his computations he actually derived the explicit form of the coefficients for Bernoulli's presentation, a slightly different version from the form usual nowadays.

Crucial progress in this discussion was achieved half a century later, due to the French mathematician, Joseph Fourier. This same Joseph Fourier, who had already taken part in the promotion of the French revolution, was the governor of Lower Egypt and secretary of the Cairo Institute, had a Chair in Mathematics in L'Ecole Polytechnique, etc. In 1811, Fourier submitted his paper on the theory of heat conduction (Mémoire sur la propagation de la chaleur) to the Paris Academy, as a candidate for the Great Prize in Mathematics (Grand Prix de Mathématiques) for the year 1812. Fourier derived the heat equation

$$\psi_t - \alpha \psi_{xx} = 0, \tag{1.4}$$

developed the method of separation of variables to solve it, and laid the foundations for what is now known as Fourier analysis. The new-born baby had its problems: although Fourier got the prize, it was accompanied by a lot of criticisms from the

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referees. The list of referees included Lagrange, Laplace, and Legendre. The list of criticisms included accusations of the absence of rigor in his analysis, stating that "the manner in which the author arrives at these equations is not exempt from difficulties and that his analysis to integrate them still leaves something to be desired on the score of generality and even rigor." A very bitter pill, accompanying the award, was the fact that Fourier's paper was not published in the Proceedings of the Academy (Mémoires de l'Académie des Sciences) till 1822, when Fourier was already the Secretary of the Academy. In all fairness, we should notice that the publication was arranged by Delambre, not by Fourier himself. But the publication did not improve the situation much, and for the last eight years of his life he still had to tackle the assaults of Biot and Poisson; the only difference from the past was that the direction of the attacks changed. In 1810, the main target of Fourier's opponents was the incorrectness of his results, while 20 years later it suddenly became the matter of priority.

Another half a century and the efforts of Cauchy, Dirichlet, Riemann, and many other great mathematicians were needed to figure out the conditions of convergency for Fourier integrals under different assumptions, necessary and sufficient conditions for various classes of (piecewise) continuous functions to have Fourier presentation, etc. Armed with our present knowledge, we can say that for the simplest possible example – a periodic function $\psi(x)$ of one variable, with period 2π and with $\int_0^{2\pi} \psi^2(x)\mathbf{d}x < \infty$ – two equivalent Fourier presentations can be given in trigonometrical and complex form:

$$\psi(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{1.5}$$

or

$$\psi(x) \sim \sum_{-\infty}^{\infty} c_n \exp(inx), \tag{1.6}$$

with coefficients a_n, b_n, c_n given below:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \psi(x)\mathbf{d}x, \tag{1.7}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \psi(x)\cos nx\mathbf{d}x, \tag{1.8}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \psi(x)\sin nx\mathbf{d}x, \tag{1.9}$$

and

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}, \tag{1.10}$$

with $n = 1, 2, 3, \ldots$, which follows from the Euler presentation

$$\exp(ix) = \cos x + i \sin x, \quad \exp(-ix) = \cos x - i \sin x. \tag{1.11}$$

Each of the series (1.5), (1.6) converges to $\psi(x)$ in the mean; if $\psi(x)$ is a continuously differentiable function, its Fourier series converges uniformly. For a function of multiple variables, similar formulas can be obtained.

One of the most important developments of Fourier analysis, from the computational point of view, was the question of whether or not it is possible to approximate an arbitrary function by a finite trigonometrical Fourier sum. The answer given by Weierstrass in 1885 was positive: for any periodic continuous function $\psi(x)$ defined on a compact and any arbitrary small number ε such that $0 < \varepsilon \ll 1$, there exists a finite Fourier sum $\mathcal{S}_N$, though not unique,

$$\mathcal{S}_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N<\infty} (a_n \cos nx + b_n \sin nx) \tag{1.12}$$

such that

$$|\psi(x) - \mathcal{S}_N| < \varepsilon \tag{1.13}$$

for any x from the definition domain of the function $\psi(x)$. This means that Fourier analysis can be used for the presentation of an arbitrary periodic function as a finite set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to the original problem or an approximation to it to whatever accuracy is desired or practical. With this theorem, Weierstrass in fact laid the sound foundations for the theory of numerical methods for solving PDEs – some 60 years before the first electronic computer was built.

Superposition principle

Notice that equations (1.1), (1.4), and many other physically relevant equations, e.g. the Laplace equation

$$\psi_{xx} + \psi_{yy} = 0, \tag{1.14}$$

are linear equations and can be written in the general form

$$\mathcal{L}[\psi] = \mathcal{F}, \tag{1.15}$$

where $\mathcal{L}$ is a linear partial differential operator (LPDO) being

$$\frac{\partial}{\partial_{tt}} - \alpha^2 \frac{\partial}{\partial_{xx}}, \quad \frac{\partial}{\partial_t} - \alpha \frac{\partial}{\partial_{xx}} \quad \text{and} \quad \frac{\partial}{\partial_{xx}} + \frac{\partial}{\partial_{yy}} \tag{1.16}$$

for equations (1.1), (1.4), and (1.14), respectively. A linear operator preserves the form of linear combinations of functions with constant coefficients, i.e.

$$\mathcal{L}\left[\sum_{j=1}^{k} a_j\psi_j\right] = \sum_{j=1}^{k} a_j\mathcal{L}[\psi_j]. \tag{1.17}$$

Also the initial and boundary conditions are usually given as linear equations on the function ψ and its derivatives (see Section 1.2 for more details). As we will see below, the principle of superposition described by (1.17) is of most importance in Fourier analysis.

Example 1: Heat transport along a bar of finite length

To show the inner mechanics of the method, let us follow Fourier and regard heat transport along a bar with length l. The temperature $\psi(x,t)$ of the bar satisfies equation (1.4) with boundary conditions taken in the form

$$\psi(0,t) = \psi(l,t) = 0, \quad t > 0, \tag{1.18}$$

and the initial conditions for the temperature along the bar

$$\psi(x,0) = \varphi(x), \quad t = 0, \ \ 0 < x < l. \tag{1.19}$$

Assume that the solution to (1.4) has the form

$$\psi(x,t) = X(x)T(t) \tag{1.20}$$

and substitute (1.20) into (1.4)

$$XT_t = \alpha X_{xx}T \ \Rightarrow \ \frac{T_t}{\alpha T} = \frac{X_{xx}}{X} \equiv \lambda \tag{1.21}$$

and obviously $\lambda \equiv$ const. This way the PDE is reduced to two ordinary differential equations (ODEs)

$$T_x = \alpha\lambda T, \quad \text{and} \quad X_{xx} = \lambda X. \tag{1.22}$$

Let λ be real, and regard cases:

Case 1: $\lambda > 0$. Solutions have the form

$$T(t) = A\exp(\alpha\lambda t) \tag{1.23}$$

and

$$X(x) = B\exp\left(x\sqrt{\lambda}\right) - C\exp\left(-x\sqrt{\lambda}\right). \tag{1.24}$$

The boundary conditions (1.18) are satisfied if and only if $X(0) = X(l) = 0$, i.e.

$$0 = X(0) = B + C, \quad 0 = X(l) = B \exp\left(l\sqrt{\lambda}\right) + C \exp\left(-l\sqrt{\lambda}\right),$$
$$\Rightarrow \exp\left(l\sqrt{\lambda}\right) + \exp\left(-l\sqrt{\lambda}\right) = 0 \Rightarrow \sinh\left(l\sqrt{\lambda}\right) = 0, \quad (1.25)$$

which is not possible.

Case 2: $\lambda = 0$. This yields

$$T_t = 0, \quad X_{xx} = 0 \Rightarrow T(t) = A, \quad X(x) = B + Cx, \quad (1.26)$$

and imposing the boundary conditions gives

$$0 = X(0) = B, \quad 0 = X(l) = Cl \Rightarrow B = C = 0, \quad (1.27)$$

i.e. also in this case there are no nontrivial solutions.

Case 3: $\lambda < 0$. Let $\lambda = -\mu^2$ with a real μ, then again the boundary conditions (1.18) yield

$$\exp(i\mu l) + \exp(-i\mu l) = 0, \quad \text{i.e.} \quad \sin(\mu l) = 0. \quad (1.28)$$

The solutions of (1.28) have the form

$$\mu l = n\pi, \quad n = \pm 1, \pm 2, \ldots \quad (1.29)$$

and correspondingly

$$T_n(t) = \exp\left(-\frac{\alpha n^2 \pi^2 t}{l^2}\right), \quad (1.30)$$
$$X_n(x) = \exp\left(\frac{in\pi x}{l}\right) - \exp\left(-\frac{in\pi x}{l}\right) = 2i \sin\left(\frac{n\pi x}{l}\right), \quad (1.31)$$

with $n = 1, 2, \ldots$. By superposition, the final form of the solution satisfying the boundary conditions reads

$$\psi(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{\alpha n^2 \pi^2 t}{l^2}\right) \sin \frac{n\pi x}{l}, \quad (1.32)$$

while the initial conditions (1.19) must have a form

$$\psi(x,0) = \varphi(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \quad \text{for} \quad 0 < x < l. \quad (1.33)$$

By imposing different boundary conditions, namely

$$\psi_x(0,t) = \psi_x(l,t) = 0, \quad t > 0, \tag{1.34}$$

instead of (1.18), another solution can be deduced

$$\psi(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(-\frac{\alpha n^2 \pi^2 t}{l^2}\right) \cos\frac{n\pi x}{l}, \tag{1.35}$$

with initial conditions given by

$$\psi(x,0) = \varphi(x) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \quad \text{for} \quad 0 < x < l. \tag{1.36}$$

∎

Resonance conditions

As we have seen above, the linearity of a differential operator is important when applying Fourier analysis. On the other hand, a great majority of physically relevant differential equations are nonlinear. In order to show that nonlinear PDEs in physics are not exotic, we follow [247] describing a class of problems that leads immediately to some nonlinear PDE. In many physical problems, we need to find some relation between two quantities, say unit density ρ of some physical entity and its unit flow ψ, so that the velocity of a flow could be defined as ψ/ρ. Both ρ and ψ are functions of space and time variables, $\rho = \rho(x,t)$ and $\psi = \psi(x,t)$. Most physical systems have some conservation law that could be written as

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho dx + \psi(x_2,t) - \psi(x_1,t) = 0 \tag{1.37}$$

for any fixed interval x_1, x_2, as shown in Fig. 1.1. And if ρ and ψ are smooth, then in the limit $x_1 \to x_2$ and the conservation law takes a form

$$\rho_t + \psi_x = 0. \tag{1.38}$$

Very often there exist some intuitive or empirical considerations allowing us to regard flow as a function of density, i.e. $\psi = F(\rho)$, which yields immediately a nonlinear PDE

$$\rho_t + c(\psi)\psi_x = 0 \tag{1.39}$$

with $c(\psi) = F_x(\rho)$. This PDE describes many various physical and technical problems, such as flood waves in rivers (wave height h plays the role of density and

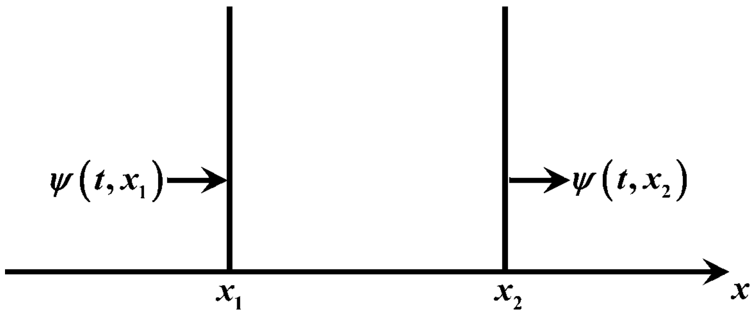

Fig. 1.1 Conservation law

$c(h)$ is the flow velocity), transport flow (ρ now is the number of cars at the unit length of a highway, ψ is the number of cars passing a line x in unit time, and the highway has no outlets), erosion of mountains, chemical exchange processes, absorption, etc.

The natural question to ask is when is it possible to linearize a given nonlinear differential equation by an appropriate invertible change of variables. It is easier to answer this question beginning with a nonlinear ODE example.

Example 2: Elimination of the quadratic term

Consider

$$\dot{\psi} = \lambda\psi + \psi^2 \tag{1.40}$$

and try to transform it into the linear form

$$\dot{\varphi} = \lambda\varphi \tag{1.41}$$

by the change of variables $\psi = \varphi + a\varphi^2$. Here a is an unknown parameter and should be chosen in such a way that the term ψ^2 in (1.40) disappears. It is easy to see that

$$\begin{gathered}
\dot{\psi} = \dot{\varphi} + 2a\varphi\dot{\varphi} = (1 + 2a\varphi)\dot{\varphi}, \ \Rightarrow \\
(1 + 2a\varphi)\dot{\varphi} = \lambda(\varphi + a\varphi^2) + (\varphi + a\varphi^2)^2 = \\
\lambda(1 + 2a\varphi)\varphi + (1 - \lambda a)\varphi^2 + p(\varphi) \ \Rightarrow \\
\dot{\varphi} = \lambda\varphi + \frac{1 - \lambda a}{1 + 2a\varphi}\varphi^2 + \frac{1}{1 + 2a\varphi}p(\varphi).
\end{gathered} \tag{1.42}$$

The use of a series presentation of $(1 + 2a\varphi)^{-1}$ over the powers of φ allows us to conclude that $(1+2a\varphi)^{-1}p(\varphi)$ contains only the terms with $\varphi^3,\ \varphi^4, \ldots$. If $\lambda \neq 0$, then the choice $a = 1/\lambda$ gives

$$\dot{\varphi} = \lambda\varphi + \frac{1}{1+2a\varphi} p(\varphi). \tag{1.43}$$

Similarly, the change of variables $\varphi = \zeta + \zeta^3$ will annihilate the term with φ^3 and so on. ∎

Moreover, a similar procedure can be applied for systems of nonlinear ODEs. Indeed, let us consider a system of nonlinear ODEs of the simple form

$$\dot{\psi}_j = \lambda_j \psi_j + \sum a_j(m)\psi^m, \quad j = 1, 2, \ldots, N \tag{1.44}$$

with multiple indexes, i.e. $m \equiv (m_1, \ldots, m_N)$, $a_j \equiv a_j(m_1, \ldots, m_N)$, $\psi^m \equiv \psi^{m_1} \cdot \ldots \psi^{m_N}$, and the minimal term in this sum is of the order ≥ 2, i.e. $m_1 + \cdots + m_N = |m| \geq 2$. Again, a simple change of variables

$$\psi_j = \varphi_j + \sum_{|m|\geq 2} b_j(m)\varphi_j^m \tag{1.45}$$

with some (not yet known) coefficients b_j, yields a new system

$$\dot{\varphi_j} = \lambda_j \varphi_j + \sum_{|m|\geq 2} c_j(m)\varphi_j^m. \tag{1.46}$$

Let us try to choose the coefficients b_j in such a way that $c_j(m) = 0\ \forall j,\ m$. If this is possible, the change of variables (1.46) will linearize the system of nonlinear ODEs (1.44). Direct substitution of (1.46) into (1.44) allows us to determine b_i only if the corresponding coefficient $-\lambda_i + \sum_{j=1}^{N}(m_j\lambda_j) \neq 0$. Otherwise, (1.46) remains nonlinear.

Definition 1. If there exist N positive integers $m_1, m_2, \ldots, m_N$, *such that*

$$\sum_{j=1}^{N} m_j \geq 2 \quad \textit{and} \quad \sum_{j=1}^{N} m_j\lambda_j - \lambda_i = 0, \tag{1.47}$$

then (1.47) are called in mathematics the ***resonance conditions*** *for the system (1.44). The number N is called the* ***order of resonance****.*

Definition 2. The change of variables (1.45) is called the ***Poincaré transformation****.*

Theorem 1 (The Poincaré theorem on linearization of the vector field). If the resonance conditions (1.47) are ***not fulfilled****, then (1.44) can be linearized by an appropriate Poincaré transformation.*

Definition 3. If the resonance conditions (1.47) are fulfilled for some set of integers $m_1, \ldots, m_N$, *then (1.46) is called* ***the normal form*** *of (1.44).*

The normal form (1.46) is equivalent to (1.44) with all nonresonant terms being eliminated by the Poincaré transformation. Usually, the general form of b_j is not known and is computed order by order. The combination of the Poincaré theorem with the Poincaré transformation yields a fact of utmost importance, which is indeed sound ground for the entire nonlinear resonance analysis:

A system of nonlinear ODEs, if not linearizable, can be transformed into normal form, with the resonance term having the smallest order.

To see the significance of this issue, let us suppose that φ in (1.42) is small, $0 < \varphi \sim \varepsilon \ll 1$, and the term with $\varphi^2 \sim \varepsilon^2$ is resonant. Then all other terms, being of order $\varepsilon^3,\ \varepsilon^4, \dots$, can be neglected and a solution to

$$\dot{\varphi} = \lambda\varphi + \frac{1-\lambda a}{1+2a\varphi}\varphi^2 \tag{1.48}$$

will give an approximate solution to (1.46), with the terms of the next order of smallness omitted. Notice that the form of equations (1.40) and (1.48) differs only by the coefficient in front of the second-order term:

$$\dot{\psi} = \lambda\psi + \psi^2 \quad \text{and} \quad \dot{\varphi} = \lambda\varphi + \frac{1-\lambda a}{1+2a\varphi}\varphi^2, \tag{1.49}$$

with $\psi = \varphi + \mathcal{O}(\varepsilon^3)$. Of course, most physically relevant equations are PDEs, not ODEs. However, the notion of resonance turned out to be so important that physical classifications of PDEs [248] have been developed based on the fact of whether or not a PDE might possess some resonance.

This classification, different from standard mathematical classification, will be presented in the Section 1.2. The next step – to proceed from a nonlinear PDE to a system of nonlinear ODEs – can be performed by a number of variation and perturbation methods, an example is given in Section 1.2. Hamiltonian formalism is briefly sketched in Section 1.3 for it allows us to simplify and standardize all these methods and obtain the universal form of dynamical equations, independent of the details of the physical system. These equations are written using canonical variables, which can be transformed back to physical variables using Fourier transformation.

This way Poincaré's approach works perfectly also for the case of PDEs, *modulo* zero denominators in (1.48). The problem known nowadays as the *small divisor problem* was regarded by Poincaré as "the fundamental problem of dynamics" [155].

This book is devoted to the study of what is happening when divisors *are* zero or small enough to cause the problem while applying asymptotical methods.