Essential Mathematical Methods for the Physical Sciences

The mathematical methods that physical scientists need for solving substantial problems in their fields of study are set out clearly and simply in this tutorial-style textbook. Students will develop problem-solving skills through hundreds of worked examples, self-test questions and homework problems. Each chapter concludes with a summary of the main procedures and results and all assumed prior knowledge is summarized in one of the appendices. Over 300 worked examples show how to use the techniques and around 100 self-test questions in the footnotes act as checkpoints to build student confidence. Nearly 400 end-of-chapter problems combine ideas from the chapter to reinforce the concepts. Hints and outline answers to the odd-numbered problems are given at the end of each chapter, with fully worked solutions to these problems given in the accompanying Student Solution Manual. Fully worked solutions to all problems, password-protected for instructors, are available at www.cambridge.org/essential.

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Essential Mathematical Methods for the Physical Sciences

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Contents

Preface xiii
Review of background topics xvi

1 Matrices and vector spaces 1

1.1 Vector spaces 2
1.2 Linear operators 5
1.3 Matrices 7
1.4 Basic matrix algebra 8
1.5 Functions of matrices 13
1.6 The transpose of a matrix 13
1.7 The complex and Hermitian conjugates of a matrix 14
1.8 The trace of a matrix 16
1.9 The determinant of a matrix 17
1.10 The inverse of a matrix 21
1.11 The rank of a matrix 25
1.12 Simultaneous linear equations 27
1.13 Special types of square matrix 36
1.14 Eigenvectors and eigenvalues 40
1.15 Determination of eigenvalues and eigenvectors 45
1.16 Change of basis and similarity transformations 49
1.17 Diagonalization of matrices 51
1.18 Quadratic and Hermitian forms 53
1.19 Normal modes 58
1.20 The summation convention 67
   Summary 68
   Problems 72
   Hints and answers 83

2 Vector calculus 87

2.1 Differentiation of vectors 87
2.2 Integration of vectors 92
2.3 Vector functions of several arguments 93
2.4 Surfaces 94
2.5 Scalar and vector fields 96
## Contents

2.6 Vector operators .................................................. 96  
2.7 Vector operator formulae ........................................ 103  
2.8 Cylindrical and spherical polar coordinates ............... 107  
2.9 General curvilinear coordinates  
  Summary .......................................................... 113  
  Problems ......................................................... 119  
  Hints and answers .............................................. 121  

3 Line, surface and volume integrals  
3.1 Line integrals ................................................... 128  
3.2 Connectivity of regions ....................................... 134  
3.3 Green’s theorem in a plane .................................... 135  
3.4 Conservative fields and potentials ......................... 138  
3.5 Surface integrals .............................................. 141  
3.6 Volume integrals ............................................. 147  
3.7 Integral forms for grad, div and curl ....................... 149  
3.8 Divergence theorem and related theorems ................. 153  
3.9 Stokes’ theorem and related theorems ..................... 158  
  Summary .......................................................... 161  
  Problems ......................................................... 163  
  Hints and answers .............................................. 168  

4 Fourier series  
4.1 The Dirichlet conditions ....................................... 170  
4.2 The Fourier coefficients ...................................... 172  
4.3 Symmetry considerations .................................... 174  
4.4 Discontinuous functions ..................................... 175  
4.5 Non-periodic functions ..................................... 176  
4.6 Integration and differentiation  
  Summary .......................................................... 179  
  Problems ......................................................... 180  
  Hints and answers .............................................. 189  

5 Integral transforms .............................................. 191  
5.1 Fourier transforms ........................................... 191  
5.2 The Dirac $\delta$-function ................................... 197  
5.3 Properties of Fourier transforms  
  Summary .......................................................... 202  
5.4 Laplace transforms .......................................... 209  
5.5 Concluding remarks  
  Summary .......................................................... 217  

Contents

Problems  219
Hints and answers  226

6 Higher-order ordinary differential equations  228
  6.1 General considerations  229
  6.2 Linear equations with constant coefficients  233
  6.3 Linear recurrence relations  237
  6.4 Laplace transform method  242
  6.5 Linear equations with variable coefficients  244
  6.6 General ordinary differential equations  258
      Summary  262
      Problems  264
      Hints and answers  271

7 Series solutions of ordinary differential equations  273
  7.1 Second-order linear ordinary differential equations  273
  7.2 Ordinary and singular points of an ODE  275
  7.3 Series solutions about an ordinary point  277
  7.4 Series solutions about a regular singular point  280
  7.5 Obtaining a second solution  286
  7.6 Polynomial solutions
      Summary  292
      Problems  293
      Hints and answers  297

8 Eigenfunction methods for differential equations  298
  8.1 Sets of functions  300
  8.2 Adjoint, self-adjoint and Hermitian operators  303
  8.3 Properties of Hermitian operators  305
  8.4 Sturm–Liouville equations  308
  8.5 Superposition of eigenfunctions: Green’s functions
      Summary  315
      Problems  316
      Hints and answers  320

9 Special functions  322
  9.1 Legendre functions  322
  9.2 Associated Legendre functions  333
  9.3 Spherical harmonics  339
  9.4 Chebyshev functions  341
## Contents

9.5 Bessel functions 347  
9.6 Spherical Bessel functions 360  
9.7 Laguerre functions 361  
9.8 Associated Laguerre functions 366  
9.9 Hermite functions 369  
9.10 The gamma function and related functions 373  
  Summary 377  
  Problems 380  
  Hints and answers 385

10 Partial differential equations 387  
10.1 Important partial differential equations 387  
10.2 General form of solution 392  
10.3 General and particular solutions 393  
10.4 The wave equation 405  
10.5 The diffusion equation 408  
10.6 Boundary conditions and the uniqueness of solutions 411  
  Summary 413  
  Problems 414  
  Hints and answers 419

11 Solution methods for PDEs 421  
11.1 Separation of variables: the general method 421  
11.2 Superposition of separated solutions 425  
11.3 Separation of variables in polar coordinates 433  
11.4 Integral transform methods 455  
11.5 Inhomogeneous problems – Green’s functions 460  
  Summary 476  
  Problems 479  
  Hints and answers 486

12 Calculus of variations 488  
12.1 The Euler–Lagrange equation 489  
12.2 Special cases 490  
12.3 Some extensions 494  
12.4 Constrained variation 496  
12.5 Physical variational principles 498  
12.6 General eigenvalue problems 501  
12.7 Estimation of eigenvalues and eigenfunctions 503  
12.8 Adjustment of parameters 506  
  Summary 507
# Contents

Problems 509  
Hints and answers 514  

## 13 Integral equations 516

13.1 Obtaining an integral equation from a differential equation 516  
13.2 Types of integral equation 517  
13.3 Operator notation and the existence of solutions 518  
13.4 Closed-form solutions 519  
13.5 Neumann series 526  
13.6 Fredholm theory 528  
13.7 Schmidt–Hilbert theory 529  
Summary 532  
Problems 534  
Hints and answers 538  

## 14 Complex variables 540

14.1 Functions of a complex variable 541  
14.2 The Cauchy–Riemann relations 543  
14.3 Power series in a complex variable 547  
14.4 Some elementary functions 549  
14.5 Multivalued functions and branch cuts 551  
14.6 Singularities and zeros of complex functions 553  
14.7 Conformal transformations 556  
14.8 Complex integrals 559  
14.9 Cauchy’s theorem 563  
14.10 Cauchy’s integral formula 566  
14.11 Taylor and Laurent series 568  
14.12 Residue theorem 573  
Summary 576  
Problems 578  
Hints and answers 580  

## 15 Applications of complex variables 582

15.1 Complex potentials 582  
15.2 Applications of conformal transformations 587  
15.3 Definite integrals using contour integration 590  
15.4 Summation of series 597  
15.5 Inverse Laplace transform 599  
15.6 Some more advanced applications 602  
Summary 605
## Contents

Problems 606  
Hints and answers 610  

### 16  Probability 612

16.1  Venn diagrams 612  
16.2  Probability 617  
16.3  Permutations and combinations 627  
16.4  Random variables and distributions 633  
16.5  Properties of distributions 638  
16.6  Functions of random variables 642  
16.7  Generating functions 646  
16.8  Important discrete distributions 654  
16.9  Important continuous distributions 666  
16.10  The central limit theorem 681  
16.11  Joint distributions 683  
16.12  Properties of joint distributions 685  
Summary 691  
Problems 695  
Hints and answers 703  

### 17  Statistics 705

17.1  Experiments, samples and populations 705  
17.2  Sample statistics 706  
17.3  Estimators and sampling distributions 713  
17.4  Some basic estimators 721  
17.5  Data modeling 730  
17.6  Hypothesis testing 735  
Summary 755  
Problems 759  
Hints and answers 764  

### A  Review of background topics 766

A.1  Arithmetic and geometry 766  
A.2  Preliminary algebra 768  
A.3  Differential calculus 770  
A.4  Integral calculus 771  
A.5  Complex numbers and hyperbolic functions 773  
A.6  Series and limits 774  
A.7  Partial differentiation 777  
A.8  Multiple integrals 778  
A.9  Vector algebra 779  
A.10  First-order ordinary differential equations 781
## Contents

B Inner products 782

C Inequalities in linear vector spaces 784

D Summation convention 786

E The Kronecker delta and Levi–Civita symbols 789

F Gram–Schmidt orthogonalization 793

G Linear least squares 795

H Footnote answers 797

Index 810
Since *Mathematical Methods for Physics and Engineering* (Cambridge: Cambridge University Press, 1998) by Riley, Hobson and Bence, hereafter denoted by *MMPE*, was first published, the range of material it covers has increased with each subsequent edition (2002 and 2006). Most of the additions have been in the form of introductory material covering polynomial equations, partial fractions, binomial expansions, coordinate geometry and a variety of basic methods of proof, though the third edition of *MMPE* also extended the range, but not the general level, of the areas to which the methods developed in the book could be applied. Recent feedback suggests that still further adjustments would be beneficial. In so far as content is concerned, the inclusion of some additional introductory material such as powers, logarithms, the sinusoidal and exponential functions, inequalities and the handling of physical dimensions, would make the starting level of the book better match that of some of its readers.

To incorporate these changes, and others to increase the user-friendliness of the text, into the current third edition of *MMPE* would inevitably produce a text that would be too ponderous for many students, to say nothing of the problems the physical production and transportation of such a large volume would entail. It is also the case that for students for whom a course on mathematical methods is not their first engagement with mathematics beyond high school level, all of the additional introductory material, as well as some of that presented in the early chapters of the original *MMPE*, would be ground already covered. For such students, typically those who have already taken two or three semesters of calculus, and perhaps an introductory course in ordinary differential equations, much of the first half of such an omnibus edition would be redundant.

For these reasons, we present under the current title, *Essential Mathematical Methods for the Physical Sciences*, an alternative edition of *MMPE*, one that focuses on the core of a putative extended third edition, omitting, except in summary form, all of the “mathematical tools” at one end, and some of the more specialized topics that appear in the third edition at the other. The emphasis is very much on developing the methods required by physical scientists before they can apply their knowledge of mathematical concepts to significant problems in their chosen fields.

For the record, we note that the more advanced topics in the third edition of *MMPE* that have fallen victim to this approach are quantum operators, tensors, group and representation theory, and numerical methods. The chapters on special functions, and the applications of complex variables have both been reduced to some extent, as have those on probability and statistics.

At the other end of the spectrum, the excised introductory material has not been altogether lost. Indeed, Appendix A of the present text consists entirely of summaries, in the style described in the penultimate paragraph of this Preface, of the material that
is presumed to have been previously studied and mastered by the student. Clearly it can be used both as a reference/reminder and as an indicator of some missing background knowledge.

One aspect that has remained constant throughout the three editions of *MMPE*, is the general style of presentation of a topic – a qualitative introduction, physically based wherever possible, followed by a more formal presentation or proof, and finished with one or two full-worked examples. This format has been well received by reviewers, and there is no reason to depart from its basic structure.

In terms of style, many physical science students appear to be more comfortable with presentations that contain significant amounts of verbal explanation and comment, rather than with a series of mathematical equations the last line of which implies “job done”. We have made changes that move the text in this direction. As is explained below, we also feel that if some of the advantages of small-group face-to-face teaching could be reflected in the written text, many students would find it beneficial.

One of the advantages of an oral approach to teaching, apparent to some extent in the lecture situation, and certainly in what are usually known as tutorials,\(^1\) is the opportunity to follow the exposition of any particular point with an immediate short, but probing, question that helps to establish whether or not the student has grasped that point. This facility is not normally available when instruction is through a written medium, without having available at least the equipment necessary to access the contents of a storage disc.

In this book we have tried to go some way towards remedying this by making a non-standard use of footnotes. Some footnotes are used in traditional ways, to add a comment or a pertinent but not essential piece of additional information, to clarify a point by restating it in slightly different terms, or to make reference to another part of the text or an external source. However, about half of the nearly 300 footnotes in this book contain a question for the reader to answer or an instruction for them to follow; neither will call for a lengthy response, but in both cases an understanding of the associated material in the text will be required. This parallels the sort of follow-up a student might have to supply orally in a small-group tutorial, after a particular aspect of their written work has been discussed.

Naturally, students should attempt to respond to footnote questions using the skills and knowledge they have acquired, re-reading the relevant text if necessary, but if they are unsure of their answer, or wish to feel the satisfaction of having their correct response confirmed, they can consult the specimen answers given in Appendix H. Equally, footnotes in the form of observations will have served their purpose when students are consistently able to say to themselves “I didn’t need that comment – I had already spotted and checked that particular point”.

One further feature of the present volume is the inclusion at the end of each chapter, just before the problems begin, of a summary of the main results of that chapter. For some areas, this takes the form of a tabulation of the various case types that may arise in the context of the chapter; this should help the student to see the parallels between situations which in the main text are presented as a consecutive series of often quite lengthy pieces of mathematical development. It should be said that in such a summary it is not possible to state every detailed condition attached to each result, and the reader should consider

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\(^{1}\) But in Cambridge are called “supervisions”!
Preface

the summaries as reminders and formulae providers, rather than as teaching text; that is the job of the main text and its footnotes.

Finally, we note, for the record, that the format and number of problems associated with the various remaining chapters have not been changed significantly, though problems based on excised topics have naturally been omitted. This means that hints or abbreviated solutions to all 200 odd-numbered problems appear in this text, and fully worked solutions of the same problems can be found in an accompanying volume, the Student Solution Manual for Essential Mathematical Methods for the Physical Sciences. Fully worked solutions to all problems, both odd- and even-numbered, are available to accredited instructors on the password-protected website www.cambridge.org/essential. Instructors wishing to have access to the website should contact solutions@cambridge.org for registration details.
Review of background topics

As explained in the Preface, this book is intended for those students who are pursuing a course in mathematical methods, but for whom it is not their first engagement with mathematics beyond high school level. Typically, such students will have already taken two or three semesters of calculus, and perhaps an introductory course in ordinary differential equations. The emphasis in the text is very much on developing the methods required by physical scientists before they can apply their knowledge of mathematical concepts to significant problems in their chosen fields; the basic mathematical “tools” that the student is presumed to have mastered are therefore not discussed in any detail.

However this introductory note and the associated appendix (Appendix A) are included both to act as a reference (or reminder) and to be an indicator of any presumed, but missing, topics in the student’s background knowledge. The appendix consists of summary pages for ten major topic areas, ranging from powers and logarithms at one extreme to first-order ordinary differential equations at the other. The style they adopt is identical to that used for the chapter summary pages in the 17 main chapters of the book. It should be noted that in such summaries it is not possible to state every detailed condition attached to each result. In the areas covered in Appendix A, there are very few subtle situations to consider, but the reader should be aware that they may exist.

Naturally, being only summaries, the various sections of the appendix will not be sufficient for the student who needs to catch up in some area, to learn the particular topics from scratch. A more elementary text will clearly be needed; *Foundation Mathematics for the Physical Sciences* written by the current authors would be one such possibility.