Algebraic Methods in Unstable Homotopy Theory

This is a comprehensive up-to-date treatment of unstable homotopy. The focus is on those methods from algebraic topology which are needed in the presentation of results, proven by Cohen, Moore, and the author, on the exponents of homotopy groups.

The author introduces various aspects of unstable homotopy theory, including: homotopy groups with coefficients; localization and completion; the Hopf invariants of Hilton, James, and Toda; Samelson products; homotopy Bockstein spectral sequences; graded Lie algebras; differential homological algebra; and the exponent theorems concerning the homotopy groups of spheres and Moore spaces.

This book is suitable for a course in unstable homotopy theory, following a first course in homotopy theory. It is also a valuable reference for both experts and graduate students wishing to enter the field.

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To the memory of my algebra teacher,

Clare DuBrock,

Tilden Technical High School,

Chicago, Illinois

Contents

Preface

What is in this book and what is not

The purpose of this book is to present those techniques of algebraic topology which are needed in the presentation of the results on the exponents of homotopy groups which were proven by Cohen, Moore, and the author. It was decided that all of the details of those techniques would be completely and honestly presented.

Homotopy groups with coefficients are fundamental to the whole enterprise and have and will be useful in other things. The 2-primary theory was not excluded but the fact that certain things are just not true for the 2-primary case reinforces the eventual restriction, more and more, to the odd primary case and finally to the case of primes greater than 3. The argument could have been made that the exact sequences of these groups related to pairs and to fibrations are all a consequence of the fundamental work of Barratt and of Puppe on cofibration sequences and can be found as a special case in the books of G. Whitehead or of E. H. Spanier. But the general theory does not handle the low dimensional cases which correspond to the fundamental group and the only way to provide an honest uniform treatment was to present the whole theory in detail. So that is what is done.

Localization has undergone a revolution in the hands of Dror Farjoun and of Bousfield. This new theory is incredibly general. It includes both the classical theory of inverting primes and of completion. It also includes exotic forms of localization related to a theorem of Haynes Miller. Some simplifications can be made if one restricts localizations to simply connected spaces or to H-spaces. It seemed to the author that not much is lost in terms of potential applications by so doing. The same is true if no appeal is made to the arcane theory of very large sets and if we restrict Dror Farjouns fundamental existence proof so that the largest thing we have to refer to is the least uncountable ordinal.

It seemed that localization should be presented in this new incarnation and that application should be made to the construction of the Hilton–Roitberg examples of H-spaces, to the loop space structures on completed spheres, and to Serre's questions about nonvanishing of infinitely many homotopy groups of a finite complex. The last application is not traditionally thought to have anything to do with localization.

The author has been told that the theory of fibrations in cubical diagrams is out of date and should be superseded by the more general theory of limits and colimits of diagrams. Spiritually the author agrees with this. But practically he does not. The cubical theory is quite useful and specific and easier to present.

The theory of Hopf invariants due to James, Hilton, and Toda was central to the proofs of the first exponent results for spheres. Since the new methods give new exponent results, why do we include these? There are several reasons. First, the 2-primary results of James have been substantially improved by Selick but they have not been superseded, and the best possible 2-primary bounds have not been found. In order to have any 2-primary bounds on the exponents for the homotopy groups of spheres, we still need James; and James and Hilton both give us the EHP sequences which are still fundamental in the computations of unstable homotopy theory. This latter reason is also applicable to Toda's work on odd primary components. He produces a useful factorization of the double suspension in the odd primary case which leads to the odd primary EHP sequences.

Samelson products in homotopy theory with coefficients are the main tool in the proofs of the exponent results of Cohen, Moore, and the author. These products give the homotopy theory of loop spaces the structure of graded Lie algebras with the exception of some unfortunate failure of the Jacobi identity at the prime 3. This theory is included here together with an important improvement on a theory of Samelson products over the loops on an H-space. This improvement makes possible a simplification of the main exponent proofs.

The homotopy and homology Bockstein spectral sequences are presented in detail with particular attention paid to products in the spectral sequence and to the convergence of the spectral sequence in the nonfinitely generated case. It occurred to the author that no book on Bockstein spectral sequences should be written without presenting Browder's results on the unboundedness of the order of the torsion in the homology of finite H-spaces. Even though these results are well presented in the paper of Browder and in the book of McCleary, the inclusion of these results is amply justified by their beauty and by the remarkable fact that this growth in the order of the torsion in homology is precisely opposite to the bounds on the order of the torsion that we find in homotopy.

The consideration of Samelson products in homotopy and their Hurewicz representation as commutators in homology makes it vital to present a general theory of graded Lie algebras and their universal enveloping algebras. Even though it is not necessary for our applications, for the first time we make no restrictions on the ground ring. It need not contain $\frac{1}{2}$. We prove the graded versions of the Poincaré–Birkhoff–Witt theorem and the related tensor product decompositions of universal enveloping algebras related to exact sequences. There is a similarity between free groups and free Lie algebras. Subalgebras of free Lie algebras are free but they may not be finitely generated even if the ambient Lie algebra is. Nonetheless, the generators of the kernels of homomorphisms can often be determined.

The actual Eilenberg–Moore spectral sequence plays almost no role in this book. But the chain model approximations that underlie this theory play an essential role and are fully presented here. We restrict our treatment to the case when the base is simply connected. This includes most applications and avoids delicate problems related to the convergence of the approximations. Particular attention is paid to products and coproducts in these models. A new innovation is the connection to the geometry of loop multiplication via an idea which is dual to an idea presented in a Cartan seminar by John Moore.

In the chapter on exponents of the homotopy of spheres and Moore spaces, most of the above finds application.

Finally, the major omission in this book on unstable homotopy theory is that there is no systematic treatment of simplicial sets even though they are used once in a while in this book. They are used to study Eilenberg–Zilber maps, the Alexander–Whitney maps, the Serre filtration, and Kan's construction of group models for loop spaces. Too bad, you can't include everything.

Prerequisites

The reader should be familiar with homology and homotopy groups. Homology groups can be found in the classic book by S. Eilenberg and N. Steenrod [44] or in many more recent books such as those of M. Greenberg and J. Harper [49], A. Dold [33], E. Spanier [123], and A. Hatcher [51]. Homotopy groups can be found in these books and also in the highly recommended books by G. Whitehead [134] and P. Selick [114].

Some introduction to homology and homotopy is essential before beginning to read this book. All of the subsequent suggestions are not essential but some knowledge of them would be useful and historically enlightening.

The books by G. Whitehead and P. Selick provide comprehensive introductions to homotopy theory and thus to the material in this book. Whitehead's book has an excellent treatment of Samelson products. Many of the properties of Samelson products were originally proved by him. But all the properties of Samelson products that we need are proved here.

Spectral sequences are much used in this book and we assume familiarity with them when we need them. The exposition of spectral sequences by Serre [116] remains a classic but there are alternative treatments in many places such as the books of S. MacLane [77], E. Spanier, G. Whitehead, and P. Selick. We regard the Serre spectral sequence as a basic tool and use it to prove many things. The survey by J. McCleary [82] provides an excellent overview of many spectral sequences, including the Eilenberg–Moore spectral sequence to which we devote much of this book.

Obstruction theory to extending maps and homotopies is a frequent tool. It is presented in the book of Whitehead. An important generalization to sections of fibre bundles is in the book of N. Steenrod [125].

Postnikov systems are used in the treatment of the Hurewicz theorem for homotopy groups with coefficients. Postnikov systems appear in the works of Serre [117, 118]. The standard references are the books of G. Whitehead and E. Spanier. The treatment in R. Mosher and M. Tangora [98] is brief and very clear.

The main books on homological algebra are two, that of H. Cartan and S. Eilenberg [23] and that of S. MacLane [77]. Cartan–Eilenberg's treatment of spectral sequences is used in this book in order to introduce products in the mod p homotopy Bockstein spectral sequence. MacLane's book is more concrete and provides an introduction to the the details of the Eilenberg–Zilber map and to the differential bar construction.

Ways to use this book

A book this long should be read in shorter segments. Many of the chapters are self-contained and can be read independently. Here are some ideas as to how the book can be broken up. Each of the paragraphs below is meant to indicate that that material can be read independently with minimal reference to the other chapters of the book.

The first chapter on homotopy groups with coefficients introduces these groups which are the homotopy group analog of homology groups with coefficients. The essence of it is captured in the Sections 1.1 through 1.7 which start with the definition and end with the mod k Hurewicz theorem. It is basic material. When combined with Sections 6.7 through 6.9 on Samelson products and with some of the material on Bockstein spectral sequences in Sections 7.1 to 7.6 it leads via Sections 9.6 and 9.7 on the cycles in differential graded Lie algebras to a proof of the existence of higher order torsion in the integral homotopy groups of an odd primary Moore space.

The second chapter on localization is completely self-contained. Sections 2.1 through 2.7 cover the most important parts of the classical localizations and completions of topological spaces. After that, the reader can choose from applications of Miller's theorem to the nonvanishing of the homotopy groups of a finite complex in Section 2.10, applications to the Hilton–Roitberg examples, or to loop structures on completions of spheres. This chapter is one of the most accessible in the book.

The short third chapter on Peterson–Stein formulas is a self-contained introduction to these formulas and also to the theory of fibred cubes which should be better known in homotopy theory. It is a quick treatment of fundamental facts about fibrations.

The fourth chapter on Hilton–Hopf invariants and the EHP sequence introduces many of the classical methods of unstable homotopy theory, for example, the James construction, the Hilton–Milnor theorem, and the James fibrations which underlie the EHP sequence. It contains a proof of James 2-primary exponent theorem for the spheres and some elementary computations of low dimensional homotopy groups. It is an introduction to some geometric ideas which are often used in the study of homotopy groups of spheres, especially of the 2-primary components.

The fifth chapter on James–Hopf and Toda–Hopf can serve as an odd primary continuation of the fourth chapter. It contains Toda's odd primary fibrations which give the odd primary EHP sequence and it contains the proof of Toda's odd primary exponent theorem for spheres. To study the odd primary components of the homotopy of spheres, Toda realized that it could be advantageous to decompose the double suspension into a composition which is different from the obvious one of composition of single suspensions.

The sixth chapter on Samelson products contains a complete treatment of Samelson products in odd primary homotopy groups. As mentioned above, it can be combined with the Bockstein spectral sequence and material on cycles in differential graded Lie algebras to prove the existence of higher order torsion in the homotopy groups of odd primary Moore spaces. Of course, the reader will need some knowledge of Chapter 1 here.

The seventh chapter on Bockstein spectral sequences contains a presentation of Browder's results on torsion in H-spaces which is completely independent of the rest of the book. It is included because of the beauty of the results and because it was the first deep use of the Bockstein spectral sequence.

Chapters 8 and 9 present the theory and applications of graded Lie algebras and their universal enveloping algebras. Particular attention is paid to free Lie algebras and their subalgebras. Although this section contains many results of purely algebraic interest, it also has geometric applications via the Lie algebras of Samelson products and via the study of loop spaces whose homology is a universal enveloping algebra. One of the applications of differential graded Lie algebras is the previously mentioned higher order torsion in the homotopy groups of odd primary Moore spaces.

Chapter 10 on differential homological algebra is the longest in the book and only does half of the theory, albeit it is the harder half. This half deals with the cobar construction of Adams and the so-called second quadrant Eilenberg–Moore spectral sequence. The presentation here emphasizes the chain models that underlie the spectral sequences and which are often more important and useful than the spectral sequences. Special emphasis is placed on the not so obvious relation of the loop multiplication to the homological algebra. It is the detailed foundation chapter for the next section on odd primary exponent theorems and the loop space decompositions which lead to them.

Chapter 11 on odd primary exponent theorems is the chapter which guides the book in its selection of topics. It defines the central current. It uses almost the whole book as background material. Nonetheless, it can be read independently if the reader is willing to use isolated parts of the book as background material. The necessary background material includes homotopy groups with coefficients, their Bockstein spectral sequences, and Samelson products in them. These are used to construct the product decomposition theorems which are the basis for the applications to exponent theorems. Localization is necessary because, without it, these decomposition theorems would not be valid. Free graded Lie algebras, their subalgebras and universal enveloping algebras are the algebraic models for the loop spaces we study and for their decomposition into topological products. It is difficult to do but it can be read without reading all of the rest of the book.

Finally, Chapter 12 is included because it is the other half of differential homological algebra, that which is used to study classifying spaces, and it is arguably the more important and useful half of the theory. It is also often the easier half and it contains the beautiful applications of Stiefel–Whitney classes to non-immersion and non-parallelizability results for real projective spaces. It would have been a shame not to include it.

Acknowledgments

The creation of this book involved many midwives.

First and foremost is John C. Moore who was my teacher and colleague and who was present at the creation of many of the ideas in this book.

Emmanuel Dror Farjoun introduced me to his theory of localization which is prominent in this book.

The interest of an audience is vital. My audiences were excellent and this is a better book because of them.

John Harper and Inga Johnson attended a semester's course on an early version of this book. There was always a bowl of candy on Inga's desk to energize the lecturer. John dutifully read every page of text, found many typographical errors, and shared his profound knowledge of homotopy theory to suggest many clarifications.

Lucía Fernández-Suárez arrranged for lectures in Braga, Portugal. Braga is a beautiful and historic city and Lucía provided an audience of her, Thomas Kahl, Lucille Vandembroucq, and Gustavo Granja to ask clarifying questions and to make intelligent comments about localization.

Lausanne, Switzerland is another beautiful and historic city. Kathryn Hess arranged for lectures there and her understanding and appreciation of the issues of differential homological algebra is unusual and very helpful.

Brayton Gray gave inspiration by his penetrating questions and insights into unstable homotopy theory.

I owe great thanks to the ones who stood by me during the event of a dissection of the aorta. Doris and John Harper, Michelle and Doug Ravenel, Fran Crawford and Joan Robinson, and Jacques Lewin were all present and accounted for.

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