Introduction to unstable homotopy theory

Computation of the homotopy groups $\pi_n(X)$ of a topological space X has played a central role in homotopy theory. And knowledge of these homotopy groups has inherent use and interest. Furthermore, the development of techniques to compute these groups has proven useful in many other contexts.

The study of homotopy groups falls into three parts.

First, there is the computation of specific homotopy groups $\pi_n(X)$ of spaces. This may be traced back to Poincaré [106] in the case $n = 1$:

Poincaré: $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ *is isomorphic to* $H_1(X)$.

Hurewicz [62] showed that, in the simply connected case, the Hurewicz homomorphism provides an isomorphism of the first nonzero $\pi_n(X)$ with the homology group $H_n(X)$ with $n \geq 1$:

Hurewicz: *If* X *is an* $n-1$ *connected space with* $n \geq 2$, *then* $\pi_n(X)$ *is isomorphic to* $H_n(X)$.

Hopf [58] discovered the remarkable fact that homotopy groups could be nonzero in dimensions higher than those of nonvanishing homology groups. He did this by using linking numbers but the modern way is to use the long exact sequence of the Hopf fibration sequence $S^1 \to S^3 \to S^2$.

Hopf: $\pi_3(S^2)$ *is isomorphic to the additive group of integers* $\mathbb{Z}$.

Computation enters the modern era with the work of Serre [116, 118] on the low dimensional homotopy groups of spheres . To this end, he introduced a localization technique which he called "classes of abelian groups." A first application was:

Serre: *If* $n \geq 1$ *and* p *is an odd prime, then the group* $\pi_{2n+2p-2}(S^{2n+1})$ *contains a summand isomorphic to* $\mathbb{Z}/p\mathbb{Z}$.

Second, there are results which relate the homotopy groups of some spaces to those of others.

Examples are product decomposition theorems such as the result of Serre which expresses the odd primary components of the homotopy groups of an even-dimensional sphere in terms of those of odd-dimensional spheres, that is:

Serre: *Localized away from 2, there is a homotopy equivalence*

$$\Omega S^{2n} \simeq S^{2n-1} \times \Omega S^{4n-1}.$$

Localization is necessary for some results but not for all. A product decomposition which requires no localization is the Hilton–Milnor theorem [54, 89, 134] which expresses the homotopy groups of a bouquet of two suspension spaces $\pi_k(\Sigma X \vee \Sigma Y)$ in terms of the homotopy groups of the constituents of the bouquet ΣX, ΣY, and of the homotopy groups of various smash products:

Hilton–Milnor: *There is a homotopy equivalence*

$$\Omega(\Sigma X \vee \Sigma Y) \simeq \Omega\Sigma X \times \Omega\Sigma(\bigvee_{j=0}^{\infty} X^{\wedge j} \wedge Y).$$

Third, Serre used his localization technique to study global properties of the homotopy groups of various spaces. What is meant by this is best made clear by giving various examples:

Serre: *For a simply connected complex with finitely many cells in each dimension, the homotopy groups are finitely generated.*

Serre: *Odd dimensional spheres have only one nonfinite homotopy group,* $\pi_{2n+1}(S^{2n+1}) = \mathbb{Z}$.

Serre: *Simply connected finite complexes with nonzero reduced homology have infinitely many nonzero homotopy groups.*

Serre [117] proved the last result by using the cohomology of Eilenberg–MacLane spaces. There is now a modern proof which uses Dror-Farjoun localization and Miller's Sullivan conjecture [83, 84].

The study of the global properties of homotopy groups was continued by James [66, 67] who introduced what are called the James–Hopf invariant maps. Using fibration sequences associated to these, James proved the following upper bound on the exponent of the 2-primary components of the homotopy groups of spheres:

James: 4^n *annihilates the 2-primary component of the homotopy groups of the sphere* S^{2n+1}.

James' result is a consequence of a more geometric result which was first formulated as a theorem about loop spaces by John Moore. For a homotopy associative H-space X and a positive integer k, let $k : X \to X$ denote the k-th power map defined by $k(x) = x^k$.

James: *Localized at 2, there is a factorization of the 4-th power map*

$$4 : \Omega^3 S^{2n+1} \to \Omega S^{2n-1} \to \Omega^3 S^{2n+1}.$$

Toda [130, 131] defined new "secondary" Hopf invariants and used these to extend James' result to odd primes p, that is:

Toda: *For an odd prime p, p^{2n} annihilates the p-primary component of the homotopy groups of the sphere S^{2n+1}.*

Or in Moore's reformulation:

Toda: *Localized at an odd prime p, there is a factorization of the p^2-**d** power map*

$$p^2 : \Omega^3 S^{2n+1} \to \Omega S^{2n-1} \to \Omega^3 S^{2n+1}.$$

No progress was made in the exponents of the primary components of homotopy groups until Selick's thesis [112].

Selick: *For p an odd prime, p annihilates the p-primary component of the homotopy groups of S^3.*

Selick's result is a consequence of the following geometric result. Let $S^3\langle 3\rangle$ denote the 3-connected cover of the 3-sphere S^3 and let $S^{2p+1}\{p\}$ denote the homotopy theoretic fibre of the degree p map $p : S^{2p+1} \to S^{2p+1}$.

Selick: *Localized at an odd prime p, $\Omega^2(S^3\langle 3\rangle)$ is a retract of $\Omega^2 S^{2p+1}\{p\}$.*

Selick's work was followed almost immediately by the work of Cohen–Moore–Neisendorfer [27, 26]. They proved that, if p is a prime greater than 3, then p^n annihilates the p-primary component of the homotopy groups of S^{2n+1}. A little later, Neisendorfer [100] overcame technical difficulties and extended this result to all odd primes.

Cohen–Moore–Neisendorfer: *Localized at an odd prime there is a factorization of the p-th power map*

$$p : \Omega^2 S^{2n+1} \to S^{2n-1} \to \Omega^2 S^{2n+1}.$$

Let $C(n)$ be the homotopy theoretic fibre of the double suspension map $\Sigma^2 : S^{2n-1} \to \Omega^2 S^{2n+1}$.

Exponent corollary: *If p is an odd prime, then p annihilates the p primary components of the homotopy groups $\pi_*(C(n))$ and p^n annihilates the p primary components of the homotopy groups $\pi_*(S^{2n+1})$.*

For odd primes, Brayton Gray [46] showed that the results of Selick and Cohen–Moore–Neisendorfer are the best possible. At the prime 2, the result of James is not the best possible but the definitive bound has not yet been found.

The main point of this book is to present the proof of the result of Cohen–Moore–Neisendorfer. We present the necessary techniques from homotopy theory, graded Lie algebras, and homological algebra. To this end, we need to develop homotopy groups with coefficients and the differential homological algebra associated to fibrations. These are applied to produce loop space decompositions which yield the above theorems.

It is useful to consider two cases of homotopy groups with coefficients, the case where the coefficients are a finitely generated abelian group and the case where the coefficients are a subgroup of the additive group of the rational numbers.

For a space X and finitely generated abelian group G, $\pi_n(X;G)$ is defined as the set of pointed homotopy classes of maps $[P^n(G), X]_*$ from a space $P^n(G)$ to X where $P^n(G)$ is a space with exactly one nonzero reduced cohomology group isomorphic to G in dimension n. This definition first occurs in the thesis of Peterson [104, 99]. These homotopy groups with coefficients are related to the classical homotopy groups by a universal coefficient sequence.

Peterson: *There is a short exact sequence*

$$0 \to \pi_n(X) \otimes G \to \pi_n(X;G) \to \mathrm{Tor}^1_{\mathbb{Z}}(\pi_{n-1}(X), G) \to 0.$$

There is a Hurewicz homomorphism to homology with coefficients

$$\phi : \pi_n(X;G) \to H_n(X;G),$$

the image of which lies in the primitive elements, and a Hurewicz theorem is true.

From this point of view, the usual or classical homotopy groups are those with coefficients $\mathbb{Z}$.

In the finitely generated case, nothing is lost by considering only the case of cyclic coefficients. If 2-torsion is avoided, Samelson products were introduced into these groups for a homotopy associative H-space X in the thesis of Neisendorfer [99]:

$$[\quad,\quad] : \pi_n(X;\mathbb{Z}/k\mathbb{Z}) \otimes \pi_m(X;\mathbb{Z}/k\mathbb{Z}) \to \pi_{m+n}(X;\mathbb{Z}/k\mathbb{Z}).$$

To construct these Samelson products, it is necessary to produce decompositions of smash products into bouquets:

$$P^n(\mathbb{Z}/p^r\mathbb{Z}) \wedge P^m(\mathbb{Z}/p^r\mathbb{Z}) \simeq P^{n+m}(\mathbb{Z}/p^r\mathbb{Z}) \vee P^{n+m-1}(\mathbb{Z}/p^r\mathbb{Z})$$

when p is an odd prime. If $p = 2$, these decompositions do not always exist and therefore there is no theory of Samelson products in homotopy groups with coefficients $\mathbb{Z}/2\mathbb{Z}$. If $p = 3$, the decompositions exist but the decompositions are not "associative" and this leads to the failure of the Jacobi identity for Samelson products in homotopy with $\mathbb{Z}/3\mathbb{Z}$ coefficients.

The Hurewicz homomorphism carries these Samelson products into graded commutators in the Pontrjagin ring,

$$\phi[\alpha, \beta] = [\phi\alpha, \phi\beta] = (\phi\alpha)(\phi\beta) - (-1)^{nm}(\phi\beta)(\phi\alpha)$$

where $n = \deg(\alpha)$ and $m = \deg(\beta)$.

Neisendorfer also introduced a homotopy Bockstein spectral sequence to study the order of torsion elements in the classical homotopy groups.

With few exceptions, the first applications of homotopy groups with coefficients will be to the simple situation where the the Hurewicz homomorphism is an isomorphism through a range. In a few cases, we will need to consider situations where the Hurewicz map is merely an epimorphism but with a kernel consisting only of Whitehead products in a range. This is all we will need to develop the theory of Samelson products in homotopy groups with coefficients, where we avoid the prime 2 and sometimes the prime 3.

For a space X and a subgroup G of the rationals, $\pi_n(X; G)$ is defined as the tensor product $\pi_n(X) \otimes G$. where , if $n = 1$, we require $\pi_n(X)$ to be abelian. Once again, these homotopy groups with coefficients are related to the classical homotopy groups by a universal coefficient sequence, there is a Hurewicz homomorphism to homology with coefficients, and a Hurewicz theorem is true. Futhermore, there are Samelson products for a homotopy associative H-space X and the Hurewicz map carries these Samelson products into graded commutators in the Pontrjagin ring.

In the special case of rational coefficients Q, the Hurewicz homomorphism satisfies a strong result of Milnor–Moore [90]:

Milnor–Moore: *If X is a connected homotopy associative H-space, then the Hurewicz map $\varphi : \pi_*(X; Q) \to H_*(X; Q)$ is an isomorphism onto the primitives of the Pontrjagin ring and there is an isomorphism*

$$H_*(X; Q) \cong U(\pi_*(X; Q))$$

where UL denotes the universal enveloping algebra of a Lie algebra L.

In practice this means that the rational homotopy groups can often be completely determined and this is one of things that makes rational homotopy groups useful.

In contrast, homotopy groups with cyclic coefficients have not been much used since they are usually as difficult to completely determine as the usual homotopy groups are. Nonetheless, some applications exist. The Hurewicz map still transforms the Samelson product into graded commutators of primitive elements in the Pontrjagin ring. This representation is far from faithful but is still nontrivial. The homotopy Bockstein spectral sequence combines with the above to give information on the order of torsion homotopy elements related to Samelson products.

Many theorems in homotopy theory depend on the computation of homology. For example, in order to prove that two spaces X and Y are homotopy equivalent, one constructs a map $f : X \to Y$ and checks that the induced map in homology is an isomorphism. If X and Y are simply connected and the isomorphism is in homology with integral coefficients, then the map f is a homotopy equivalence. In general, when the isomorphism is in homology with coefficients, then the map f is some sort of local equivalence. For example, with rational coefficients, we get rational equivalences, with coefficients integers $\mathbb{Z}_{(p)}$ localized at a prime p, we get equivalences localized at p, and with $\mathbb{Z}/p\mathbb{Z}$ coeffients, we get equivalences of completions at p. The theorem of Serre, $\Omega S^{2n} \simeq S^{2n-1} \times \Omega S^{4n-1}$ localized away from 2, and the Hilton–Milnor theorem,

$$\Omega(\Sigma X \vee \Sigma Y) \simeq \Omega\Sigma X \times \Omega\Sigma\left(\bigvee_{j=0}^{\infty} X^{\wedge j} \wedge Y\right),$$

are proved in this way. A central theme of this book will be such decompositions of loop spaces.

For us, the most basic homological computation is the homology of the loops on the suspension of a connected space:

Bott–Samelson [13]: *If X is connected and the reduced homology of $\overline{H}_*(X;R)$ is free over a coefficient ring R, then there is an isomorphism of algebras*

$$T(\overline{H}_*(X;R)) \to H_*(\Omega\Sigma X;R)$$

where $T(V)$ denotes the tensor algebra generated by a module V.

Let $L(V)$ be the free graded Lie algebra generated by V. The observation that $T(V)$ is isomorphic to the universal enveloping algebra $UL(V)$ has topological consequences based on the following simple fact:

Tensor decomposition: *If $0 \to L_1 \to L_2 \to L_3 \to 0$ is a short exact sequence of graded Lie algebras which are free as R modules, then there is an isomorphism*

$$UL_2 \cong UL_1 \otimes UL_3.$$

Suppose we want to construct a homotopy equivalence of H-spaces $X \times Y \to \mathbb{Z}$ and suppose that we compute

$$H_*(X;R) = UL_1, \quad H_*(Y;R) = UL_3, \quad \textit{and} \quad H_*(\mathbb{Z};R) = UL_2.$$

Suppose also that we can construct maps $g : X \to \mathbb{Z}$ and $h : Y \to \mathbb{Z}$ such that the product $f = \mu \circ (g \times h) : X \times Y \to \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ induces a homology isomorphism (where $\mu : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is the multiplication of $\mathbb{Z}$). Then we have an equivalence localized in the sense that is appropriate to the coefficients.

Here is an example. Let $L(x_\alpha)$ denote the free graded Lie algebra generated by the set $\{x_\alpha\}$). Let $\langle x_\alpha \rangle$ denote the abelianization, that is, the free module generated by the set with all Lie brackets zero. If we localize away from 2 and x is an odd degree element, then we have a short exact sequence

$$0 \to \langle [x,x] \rangle \to L(x) \to \langle x \rangle \to 0$$

and isomorphisms

$$H_*(\Omega S^{4n-1}) \cong U(\langle [x,x] \rangle), \;\; H_*(S^{2n-1}) \cong U(\langle x \rangle),$$
$$H_*(\Omega S^{2n}) \cong U(L(x))).$$

This leads to the result of Serre: $\Omega S^{2n} \simeq S^{2n-1} \times \Omega S^{4n-1}$ localized away from 2. Thus, Serre's result is essentially a consequence of just the Bott–Samelson theorem and the tensor decomposition of universal enveloping algebras.

Consider the following additional facts concerning Lie algebras [27]:

Free subalgebras: *If L is a free graded Lie algebra and K is a subalgebra which is a split summand as an R-module, then K is a free graded Lie algebra.*

Kernel theorem: *If K is the kernel of the natural map $L(V \oplus W) \to L(V)$ of free graded Lie algebras, then K is isomorphic to the free graded Lie algebra*

$$L\left(\bigoplus_{j=0}^{\infty} V^{\otimes j} \otimes W\right)$$

where $V^{\otimes j} = V \otimes V \otimes \cdots \otimes V$, with j factors.

A direct consequence is the Hilton–Milnor theorem,

$$\Omega(\Sigma X \vee \Sigma Y) \simeq \Omega\Sigma X \times \Omega\Sigma\left(\bigvee_{j=0}^{\infty} X^{\wedge j} \wedge Y\right).$$

In order to study torsion at a prime p, it is useful to consider the Bockstein differentials in homology with mod p coefficients. This leads to consideration of differential graded Lie algebras.

For example, let $P^n(p^r) = S^{n-1} \cup_{p^r} e^n$ be the space obtained by attaching an n-cell to an $n-1$-sphere by a map of degree p^r. Then $H_*(P^n(p^r); \mathbb{Z}/p\mathbb{Z}) = \langle u, v \rangle$ with $\deg(v) = n$ and $\deg(u) = n-1$. The r-th Bockstein differential is given by $\beta^r(v) = u, \beta^r(u) = 0$. Thus, the Bott–Samelson theorem gives isomorphisms of differential Hopf algebras

$$H_*(\Omega\Sigma P^n(p^r); \mathbb{Z}/p\mathbb{Z}) \cong T(u,v) \cong UL(u,v)$$

where $L = L(u,v)$ is a differential Lie algebra which is a free Lie algebra. Any algebraic constructions with topological implications must be compatible with these Bockstein differentials. For example, the abelianization of L is $\langle u, v \rangle$.

This is compatible with differentials, leads to the short exact sequence of differential Lie algebras

$$0 \to [L,L] \to L \to \langle u, v \rangle \to 0,$$

and the tensor decomposition of universal enveloping algebras

$$H_*(\Omega\Sigma P^n(p^r);\mathbb{Z}/p\mathbb{Z}) \cong UL \cong U(\langle u,v\rangle)\otimes U([L,L]).$$

But this tensor decomposition can only be realized by a product decomposition of $\Omega\Sigma P^n(p^r)$ when p and n are odd. If we set $n-1=2m$, then we can prove [27]:

Cohen–Moore–Neisendorfer: *If p is an odd prime and $m\geq 1$, then there is a homotopy equivalence*

$$\Omega P^{2m+2}(p^r) \simeq S^{2m+1}\{p^r\}\times\Omega\left(\bigvee_{j=0}^{\infty} P^{2m+2mj+1}(p^r)\right)$$

where $S^{2m+1}\{p^r\}$ is the homotopy theoretic fibre of the degree p^r map $p^r: S^{2m+1}\to S^{2m+1}$.

The restriction to odd primes in the above is the result of the nonexistence of a suitable theory of Samelson products in homotopy groups with 2-primary coefficients.

One reason for the above parity restriction is as follows: Suppose the coefficient ring is $\mathbb{Z}/p\mathbb{Z}$ with p an odd prime. Only when n is odd (so that μ has even dimension and ν has odd dimension) can we write that

$$[L,L] = L(\mathrm{ad}^j(u)([v,v]),\mathrm{ad}^j(u)([u,v]))_{j\geq 0} =$$

the free Lie algebra on infinitely many generators with r-th Bockstein differential given by $\beta^r(\mathrm{ad}^j(u)([v,v])) = 2\mathrm{ad}^j(u)([u,v]$ for $j\geq 0$. In this case, the module of generators of $[L,L]$ is acylic with respect to the Bockstein differential and it is possible that the universal enveloping algebra U([L,L]) represents the homology of the loop space on a bouquet of Moore spaces. In fact, the isomorphisms of differential algebras

$$H_*(S^{2m+1}\{p^r\};\mathbb{Z}/p\mathbb{Z}) \cong U(\langle u,v\rangle),$$

$$H_*(\Omega\left(\bigvee_{j=0}^{\infty} P^{2m+2mj+1}(p^r)\right);\mathbb{Z}/p\mathbb{Z}) \cong U([L,L]),$$

$$H_*(\Omega P^{2m+2}(p^r);\mathbb{Z}/p\mathbb{Z}) \cong UL$$

then lead to the above product decomposition for $\Omega P^{2m+2}(p^r)$.

There is no analogous product decomposition for $\Omega P^{2m+1}(p^r)$. The situation is much more complicated because of the fact that $[L,L]$ does not have an acyclic module of generators when $L=L(u,v)$ with $\deg(u)$ odd and $\deg(v)$ even. To go further we need to study the homology $H(L,\beta^r)$.

Let x be an even degree element in a differential graded Lie algebra over the ring $\mathbb{Z}/p\mathbb{Z}$ with p an odd prime, let d denote the differential, and for $k \geq 1$ define new elements

$$\tau_k(x) = \mathrm{ad}^{p^k-1}(x)(dx)$$

$$\sigma_k(x) = \frac{1}{2}\sum_{j=1}^{p^k-1} p^{-1}(j, p^k - j)[\mathrm{ad}^{j-1}(x)(dx), \mathrm{ad}^{p^k-j-1}(x)(dx)]$$

where $(a, b) = \frac{(a+b)!}{(a!)(b!)}$ is the binomial coefficient. These elements are cycles, $d(\tau_k(x)) = 0, d(\sigma_k(x)) = 0$, and they determine the homology of the above L via the following proposition.

Homology of free Lie algebras with acyclic generators: *Let $L(V)$ be a free graded Lie algebra over the ring $\mathbb{Z}/p\mathbb{Z}$ with p an odd prime and with a differential d such that $d(V) \subseteq V$ and $H(V, d) = 0$. Write*

$$L(V) = H(L(V), d) \oplus K$$

where K is acyclic. If K has a basis $x_\alpha, dx_\alpha, y_\beta, dy_\beta$ with $\deg(x_\alpha)$ even and $\deg(y_\beta)$ odd, then $H(L(V), d)$ has a basis represented by the cycles $\tau_k(x_\alpha)$, $\sigma_k(x_\alpha)$ with $k \geq 1$.

This proposition has two main applications. The first application is to a decomposition theorem which leads to the determination of the odd primary exponents of the homotopy groups of spheres.

Decomposition theorem: *Let p be an odd prime and let $F^{2n+1}\{p^r\}$ be the homotopy theoretic fibre of the natural map $P^{2n+1}(p^r) \to S^{2n+1}$ which pinches the bottom $2n$-cell to a point. Localized at p, there is a homotopy equivalence*

$$\Omega F^{2n+1}\{p^r\} \simeq S^{2n-1} \times \prod_{k=1}^{\infty} S^{2p^k n-1}\{p^{r+1}\} \times \Omega\Sigma \bigvee_\alpha P^{n_\alpha}(p^r)$$

where

$$\bigvee_\alpha P^{n_\alpha}(p^r)$$

is an infinite bouquet of mod p^r Moore spaces.

The second application is to the existence of higher order torsion in the homotopy groups of odd primary Moore spaces:

Higher order torsion: *If p is an odd prime and $n \geq 1$, then for all $k \geq 1$ the homotopy groups $\pi_{2p^k n-1}(P^{2n+1})$ contain a summand isomorphic to $\mathbb{Z}/p^{r+1}\mathbb{Z}$.*

The following decomposition theorem is valid:

Cohen–Moore–Neisendorfer: *If p is an odd prime and $m \geq 1$, then there is a homotopy equivalence*

$$\Omega P^{2m+1}(p^r) \simeq T^{2m+1}\{p^r\} \times \Omega\Sigma \bigvee_{\alpha} P^{n_\alpha}(p^r)$$

where there is a fibration sequence

$$C(n) \times \prod_{k=1}^{\infty} S^{2p^k n-1}\{p^{r+1}\} \to T^{2m+1}\{p^r\} \to S^{2n+1}\{p^r\}.$$

A corollary of these decomposition theorems is [28]:

Cohen–Moore–Neisendorfer: *If p is an odd prime and $n \geq 3$, then p^{2r+1} annihilates the homotopy groups $\pi_*(P^n(p^r))$.*

In fact the best possible result is [102]:

Neisendorfer: *If p is an odd prime and $n \geq 3$, then p^{r+1} annihilates the homotopy groups $\pi_*(P^n(p^r))$.*