# Brownian Motion

This eagerly awaited textbook offers a broad and deep exposition of Brownian motion. Extensively class tested, it leads the reader from the basics to the latest research in the area.

Starting with the construction of Brownian motion, the book then proceeds to sample path properties such as continuity and nowhere differentiability. Notions of fractal dimension are introduced early and are used throughout the book to describe fine properties of Brownian paths. The relation of Brownian motion and random walk is explored from several viewpoints, including a development of the theory of Brownian local times from random walk embeddings. Stochastic integration is introduced as a tool, and an accessible treatment of the potential theory of Brownian motion clears the path for an extensive treatment of intersections of Brownian paths. An investigation of exceptional points on the Brownian path and an appendix on SLE processes, by Oded Schramm and Wendelin Werner, lead directly to recent research themes.

'This splendid account of the modern theory of Brownian motion puts special emphasis on sample path properties and connections with harmonic functions and potential theory, without omitting such important topics as stochastic integration, local times or relations with random walk. The most significant properties of Brownian motion are derived via powerful and elegant methods. This book, which fills a gap in the existing literature, will be of interest both to the beginner, for the clarity of exposition and the judicious choice of topics, and to the specialist, who will find neat approaches to many classical results and to some more recent ones. This beautiful book will soon become a must for anybody who is interested in Brownian motion and its applications.'

*Jean-François Le Gall, Université Paris 11 (Paris-Sud, Orsay)*

'*Brownian Motion* by Mörters and Peres, a modern and attractive account of one of the central topics of probability theory, will serve both as an accessible introduction at the level of a Master's course and as a work of reference for fine properties of Brownian paths. The unique focus of the book on Brownian motion gives it a satisfying concreteness and allows a rapid approach to some deep results.

The introductory chapters, besides providing a careful account of the theory, offer some helpful points of orientation towards an intuitive and mature grasp of the subject matter. The authors have made many contributions to our understanding of path properties, fractal dimensions and potential theory for Brownian motion, and this expertise is evident in the later chapters of the book. I particularly liked the marking of the 'leaves' of the theory by stars, not only because this offers a chance to skip on, but also because these are often the high points of our present knowledge.'

*James Norris, University of Cambridge*

'This excellent book does a beautiful job of covering a good deal of the theory of Brownian motion in a very user-friendly fashion. The approach is hands-on which makes it an attractive book for a first course on the subject. It also contains topics not usually covered, such as the 'intersection-equivalence' approach to multiple points as well as the study of slow and fast points. Other highlights include detailed connections with random fractals and a short overview of the connections with SLE. I highly recommend it.'

*Jeff Steif, Chalmers University of Technology*

# CAMBRIDGE SERIES IN STATISTICAL AND PROBABILISTIC MATHEMATICS



This series of high-quality upper-division textbooks and expository monographs covers all aspects of stochastic applicable mathematics. The topics range from pure and applied statistics to probability theory, operations research, optimisation and mathematical programming. The books contain clear presentations of new developments in the field and also of the state of the art in classical methods. While emphasising rigorous treatment of theoretical methods, the books also contain applications and discussions of new techniques made possible by advances in computational practice.

A complete list of books in the series can be found at
http://www.cambridge.org/uk/series/sSeries.asp?code=CSPM
Recent titles include the following:

5. *Bayesian Methods,* by Thomas Leonard and John S. J. Hsu
6. *Empirical Processes in M-Estimation,* by Sara van de Geer
7. *Numerical Methods of Statistics,* by John F. Monahan
8. *A User's Guide to Measure Theoretic Probability,* by David Pollard
9. *The Estimation and Tracking of Frequency,* by B. G. Quinn and E. J. Hannan
10. *Data Analysis and Graphics Using R,* by John Maindonald and John Braun
11. *Statistical Models,* by A. C. Davison
12. *Semiparametric Regression,* by D. Ruppert, M. P. Wand and R. J. Carroll
13. *Exercises in Probability,* by Loïc Chaumont and Marc Yor
14. *Statistical Analysis of Stochastic Processes in Time,* by J. K. Lindsey
15. *Measure Theory and Filtering,* by Lakhdar Aggoun and Robert Elliott
16. *Essentials of Statistical Inference,* by G. A. Young and R. L. Smith
17. *Elements of Distribution Theory,* by Thomas A. Severini
18. *Statistical Mechanics of Disordered Systems,* by Anton Bovier
19. *The Coordinate-free Approach to Linear Models,* by Michael J. Wichura
20. *Random Graph Dynamics,* by Rick Durrett
21. *Networks,* by Peter Whittle
22. *Saddlepoint Approximations with Applications,* by Ronald W. Butler
23. *Applied Asymptotics,* by A. R. Brazzale, A. C. Davison and N. Reid
24. *Random Networks for Communication,* by Massimo Franceschetti and Ronald Meester
25. *Design of Comparative Experiments,* by R. A. Bailey
26. *Symmetry Studies,* by Marlos A. G. Viana
27. *Model Selection and Model Averaging,* by Gerda Claeskens and Nils Lid Hjort
28. *Bayesian Nonparametrics,* edited by Nils Lid Hjort *et al.*
29. *From Finite Sample to Asymptotic Methods in Statistics,* by Pranab K. Sen, Julio M. Singer and Antonio C. Pedrosa de Lima

# Brownian Motion

PETER MÖRTERS AND YUVAL PERES

*with an appendix by Oded Schramm and Wendelin Werner*

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521760188



First published 2010

Printed in the United Kingdom at the University Press, Cambridge

*A catalogue record for this publication is available from the British Library*

ISBN 978-0-521-76018-8 Hardback

# Contents

# Preface

The aim of this book is to introduce Brownian motion as central object of probability theory and discuss its properties, putting particular emphasis on sample path properties. Our hope is to capture as much as possible the spirit of Paul Lévy's investigations on Brownian motion, by moving quickly to the fascinating features of the Brownian motion process, and filling in more and more details into the picture as we move along.

Inevitably, while exploring the nature of Brownian paths one encounters a great variety of other subjects: Hausdorff dimension serves from early on in the book as a tool to quantify subtle features of Brownian paths, stochastic integrals helps us to get to the core of the invariance properties of Brownian motion, and potential theory is developed to enable us to control the probability the Brownian motion hits a given set.

An important idea of this book is to make it as *interactive* as possible and therefore we have included more than 100 exercises collected at the end of each of the ten chapters. Exercises marked with the symbol S have either a hint, a reference to a solution, or a full solution given at the end of the book. We have also marked some theorems with a star to indicate that the results will not be used in the remainder of the book and may be skipped on first reading. At the end of the book we have given a short list of selected open research problems dealing with the material of the book.

This book grew out of lectures given by Yuval Peres at the Statistics Department, University of California, Berkeley in Spring 1998. We are grateful to the students who attended the course and wrote the first draft of the notes: Diego Garcia, Yoram Gat, Diogo A. Gomes, Charles Holton, Frédéric Latrémolière, Wei Li, Ben Morris, Jason Schweinsberg, Bálint Virág, Ye Xia and Xiaowen Zhou. The first draft of these notes, about 80 pages in volume, was edited by Bálint Virág and Elchanan Mossel and at this stage corrections were made by Serban Nacu and Yimin Xiao. The notes were distributed via the internet and turned out to be very popular — this demand motivated us to expand these notes to a full book hopefully retaining the character of the original notes.

Peter Mörters lectured on the topics of this book in the Graduate School in Mathematical Sciences at the University of Bath in Autumn 2003, thanks are due to the audience, and in particular to Alex Cox and Pascal Vogt, for their contributions. Yuval Peres thanks

Pertti Mattila for the invitation to lecture on this material at the joint summer school in Jyväskyla, August 1999, and Peter Mörters thanks Michael Scheutzow for the invitation to lecture at the Berlin graduate school in probability in Stralsund, April 2003.

When it became clear that the new developments around the stochastic Loewner evolution would open a new chapter in the story of Brownian motion we discussed the inclusion of a chapter on this topic. Realising that doing this rigorously in detail would go beyond the scope of this book, we asked Oded Schramm to provide an appendix describing the new developments in a less formal manner. Oded agreed and immediately started designing the appendix, but his work was cut short by his tragic and premature death in 2008. We are very grateful that Wendelin Werner accepted the task of completing this appendix at very short notice.

Several people read drafts of the book at various stages, supplied us with helpful lists of corrections, and suggested or tested exercises and references. We thank Anselm Adelmann, Tonci Antunovic, Christian Bartsch, Noam Berger, Jian Ding, Uta Freiberg, Nina Gantert, Subhroshekhar Gosh, Ben Hough, Davar Khoshnevisan, Richard Kiefer, Achim Klenke, Michael Kochler, Manjunath Krishnapur, David Levin, Nathan Levy, Arjun Malhotra, Jason Miller, Asaf Nachmias, Weiyang Ning, Marcel Ortgiese, Ron Peled, Jim Pitman, Michael Scheutzow, Perla Sousi, Jeff Steif, Kamil Szczegot, Ran Tessler, Hermann Thorisson, and Brigitta Vermesi.

We also thank several people who have contributed pictures, namely Ben Hough, Marcel Ortgiese, Yelena Shvets and David Wilson. The cover shows a planar Brownian motion with points coloured according to the occupation measure of a small neighbourhood, we thank Raissa d'Souza for providing the picture.

Peter Mörters
Yuval Peres

# Frequently used notation

**Numbers:**

| | |
|---|---|
| $\lceil x \rceil$ | the smallest integer bigger or equal to $x$ |
| $\lfloor x \rfloor$ | the largest integer smaller or equal to $x$ |
| $\mathfrak{Re}(z), \mathfrak{Im}(z)$ | the real, resp. imaginary, part of the complex number $z$ |
| $\mathrm{i}$ | the imaginary unit |

**Topology of Euclidean space $\mathbb{R}^d$:**

| | |
|---|---|
| $\mathbb{R}^d$ | Euclidean space consisting of all column vectors $x = (x_1, \ldots, x_d)^{\mathrm{T}}$ |
| $\lvert \cdot \rvert$ | Euclidean norm $\lvert x \rvert = \sqrt{\sum_{i=1}^{d} x_i^2}$ |
| $\mathcal{B}(x, r)$ | the open ball of radius $r > 0$ centred in $x \in \mathbb{R}^d$,<br>i.e. $\mathcal{B}(x,r) = \{y \in \mathbb{R}^d : \lvert x - y \rvert < r\}$ |
| $\overline{U}$ | closure of the set $U \subset \mathbb{R}^d$ |
| $\partial U$ | boundary of the set $U \subset \mathbb{R}^d$ |
| $\mathfrak{B}(A)$ | the collection of all Borel subsets of $A \subset \mathbb{R}^d$ |

**Binary relations:**

| | |
|---|---|
| $a \wedge b$ | the minimum of $a$ and $b$ |
| $a \vee b$ | the maximum of $a$ and $b$ |
| $X \stackrel{\mathrm{d}}{=} Y$ | the random variables $X$ and $Y$ have the same distribution |
| $X_n \stackrel{\mathrm{d}}{\to} X$ | the random variables $X_n$ converge to $X$ in distribution,<br>see Section 12.1 in the appendix |
| $a(n) \asymp b(n)$ | the ratio of the two sides is bounded from above and below<br>by positive constants that do not depend on $n$ |
| $a(n) \sim b(n)$ | the ratio of the two sides converges to one |

**Vectors, functions, and measures:**

| | |
|---|---|
| $I_d$ | $d \times d$ identity matrix |
| $1_A$ | indicator function with $1_A(x) = 1$ if $x \in A$ and 0 otherwise |

| | |
|---|---|
| $\delta_x$ | Dirac measure with mass concentrated on $x$, i.e. $\delta_x(A) = 1$ if $x \in A$ and 0 otherwise |
| $f^+$ | the positive part of the function $f$, i.e. $f^+(x) = f(x) \vee 0$ |
| $f^-$ | the negative part of the function $f$, i.e. $f^-(x) = -(f(x) \wedge 0)$ |
| $\mathcal{L}_d$ or $\mathcal{L}$ | Lebesgue measure on $\mathbb{R}^d$ |
| $\sigma_{x,r}$ | $(d-1)$-dimensional surface measure on $\partial\mathcal{B}(x,r) \subset \mathbb{R}^d$ if $x = 0$, $r = 1$ we also write $\sigma = \sigma_{0,1}$ |
| $\varpi_{x,r}$ | uniform distribution on $\partial\mathcal{B}(x,r)$, $\varpi_{x,r} = \frac{\sigma_{x,r}}{\sigma_{x,r}(\partial\mathcal{B}(x,r))}$, if $x = 0$, $r = 1$ we also write $\varpi = \varpi_{0,1}$ |

**Function spaces:**

| | |
|---|---|
| $\mathbf{C}(K)$ | the topological space of all continuous functions on the compact $K \subset \mathbb{R}^d$, equipped with the supremum norm $\|f\| = \sup_{x\in K} \|f(x)\|$ |
| $\mathbf{L}^p(\mu)$ | the Banach space of equivalence classes of functions $f$ with finite $\mathbf{L}^p$-norm $\|f\|_p = \Big(\int f^p\, d\mu\Big)^{1/p}$. If $\mu = \mathcal{L}_{\|K}$ we write $\mathbf{L}^p(K)$. |
| $\mathbf{D}[0,1]$ | the Dirichlet space consisting of functions $F \in \mathbf{C}[0,1]$ such that for some $f \in \mathbf{L}^2[0,1]$ and all $t \in [0,1]$ we have $F(t) = \int_0^t f(s)\, ds$. |

**Probability measures and $\sigma$-algebras:**

| | |
|---|---|
| $\mathbb{P}_x$ | a probability measure on a measure space $(\Omega, \mathcal{A})$ such that the process $\{B(t) \colon t \geqslant 0\}$ is a Brownian motion started in $x$ |
| $\mathbb{E}_x$ | the expectation associated with $\mathbb{P}_x$ |
| $\mathfrak{p}(t,x,y)$ | the transition density of Brownian motion $\mathbb{P}_x\{B(t) \in A\} = \int_A \mathfrak{p}(t,x,y)\, dy$ |
| $\mathcal{F}^0(t)$ | the smallest $\sigma$-algebra that makes $\{B(s) \colon 0 \leqslant s \leqslant t\}$ measurable |
| $\mathcal{F}^+(t)$ | the right-continuous augmentation $\mathcal{F}^+(t) = \bigcap_{s>t} \mathcal{F}^0(s)$. |

**Stopping times:**

For any Borel sets $A_1, A_2, \ldots \subset \mathbb{R}^d$ and a Brownian motion $B \colon [0,\infty) \to \mathbb{R}^d$,

$$\tau(A_1) := \inf\{t \geqslant 0 \colon\ B(t) \in A_1\}, \qquad \text{the entry time into } A_1,$$

$$\tau(A_1,\ldots,A_n) := \begin{cases} \inf\{t \geqslant \tau(A_1,\ldots,A_{n-1}) \colon B(t) \in A_n\}, & \text{if } \tau(A_1,\ldots,A_{n-1}) < \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

the time to enter $A_1$ and then $A_2$ and so on until $A_n$.

**Systems of subsets in $\mathbb{R}^d$:**

For any fixed $d$-dimensional unit cube $\mathsf{Cube} = x + [0,1]^d$ we denote:

- $\mathfrak{D}_k$ family of all half-open dyadic subcubes $D = x + \prod_{i=1}^d \big[k_i 2^{-k}, (k_i+1)2^{-k}\big) \subset \mathbb{R}^d$, $k_i \in \{0, \dots, 2^k - 1\}$, of side length $2^{-k}$
- $\mathfrak{D}$ all half-open dyadic cubes $\mathfrak{D} = \bigcup_{k=0}^\infty \mathfrak{D}_k$ in Cube
- $\mathfrak{C}_k$ family of all compact dyadic subcubes $D = x + \prod_{i=1}^d \big[k_i 2^{-k}, (k_i+1)2^{-k}\big] \subset \mathbb{R}^d$, $k_i \in \{0, \dots, 2^k - 1\}$, of side length $2^{-k}$
- $\mathfrak{C}$ all compact dyadic cubes $\mathfrak{C} = \bigcup_{k=0}^\infty \mathfrak{C}_k$ in Cube.

**Potential theory:**

For a metric space $(E, \rho)$ and mass distribution $\mu$ on $E$:

- $\phi_\alpha(x)$ the $\alpha$-potential of a point $x \in E$ defined as $\phi_\alpha(x) = \int \frac{d\mu(y)}{\rho(x,y)^\alpha}$,
- $I_\alpha(\mu)$ the $\alpha$-energy of the measure $\mu$ defined as $I_\alpha(\mu) = \iint \frac{d\mu(x)\, d\mu(y)}{\rho(x,y)^\alpha}$,
- $\mathrm{Cap}_\alpha(E)$ the $\alpha$-capacity of $E$ defined as $\mathrm{Cap}_\alpha(E) = \sup\{I_\alpha(\mu)^{-1} \colon \mu(E) = 1\}$.

For a general kernel $K \colon E \times E \to [0, \infty]$:

- $U_\mu(x)$ the potential of $\mu$ at $x$ defined as $U_\mu(x) = \int K(x,y)\, d\mu(y)$,
- $I_K(\mu)$ $K$-energy of $\mu$ defined as $I_K(\mu) = \iint K(x,y)\, d\mu(x)\, d\mu(y)$,
- $\mathrm{Cap}_K(E)$ $K$-capacity of $E$ defined as $\mathrm{Cap}_K(E) = \sup\{I_K(\mu)^{-1} \colon \mu(E) = 1\}$.

If $K(x,y) = f(\rho(x,y))$ we also write:

- $I_f(\mu)$ instead of $I_K(\mu)$,
- $\mathrm{Cap}_f(E)$ instead of $\mathrm{Cap}_K(E)$.

**Sets and processes associated with Brownian motion:**

For a linear Brownian motion $\{B(t) \colon t \geqslant 0\}$:

- $\{M(t) \colon t \geqslant 0\}$ the maximum process defined by $M(t) = \sup_{s \leqslant t} B(s)$,
- Rec the set of record points $\{t \geqslant 0 \colon B(t) = M(t)\}$,
- Zeros the set of zeros $\{t \geqslant 0 \colon B(t) = 0\}$.

For a Brownian motion $\{B(t) \colon t \geqslant 0\}$ in $\mathbb{R}^d$ for $d \geqslant 1$:

- $\mathsf{Graph}(A)$ the graph $\{(t, B(t)) \colon t \in A\} \subset \mathbb{R}^{d+1}$,
- $\mathsf{Range}(A)$ the range $\{B(t) \colon t \in A\} \subset \mathbb{R}^d$.

Occasionally these notions are used for functions $f \colon [0,\infty) \to \mathbb{R}^d$ which are not necessarily Brownian sample paths, which we indicate by appending a subindex $f$ to the notion.