CHAPTER ONE

Hexaflexagons

FLEXAGONS ARE PAPER POLYGONS, folded from straight or crooked strips of paper, which have the fascinating property of changing their faces when they are "flexed." Had it not been for the trivial circumstance that British and American notebook paper are not the same size, flexagons might still be undiscovered, and a number of top-flight mathematicians would have been denied the pleasure of analyzing their curious structures.

It all began in the fall of 1939. Arthur H. Stone, a 23-year-old graduate student from England, in residence at Princeton University on a mathematics fellowship, had just trimmed an inch from his American notebook sheets to make them fit his English binder. For amusement he began to fold the trimmed-off strips of paper in various ways, and one of the figures he made turned out to be particularly intriguing. He had folded the strip diagonally at three places and joined the ends so that it made a hexagon (see Figure 1). When he pinched two adjacent triangles together and pushed the opposite corner of the hexagon toward the center, the hexagon would open out again, like a budding flower, and show a completely new face. If, for instance, the top and bottom faces of the original hexagon were painted different colors, the new faces would come up blank and one of the colored faces would disappear!

This structure, the first flexagon to be discovered, has three faces. Stone did some thinking about it overnight and, on the following day, confirmed his belief (arrived at by pure cerebration) that a more complicated hexagonal model could be folded with six faces instead of only three. At this point, Stone found the structure so interesting that he showed his paper models to friends in the graduate school.

1

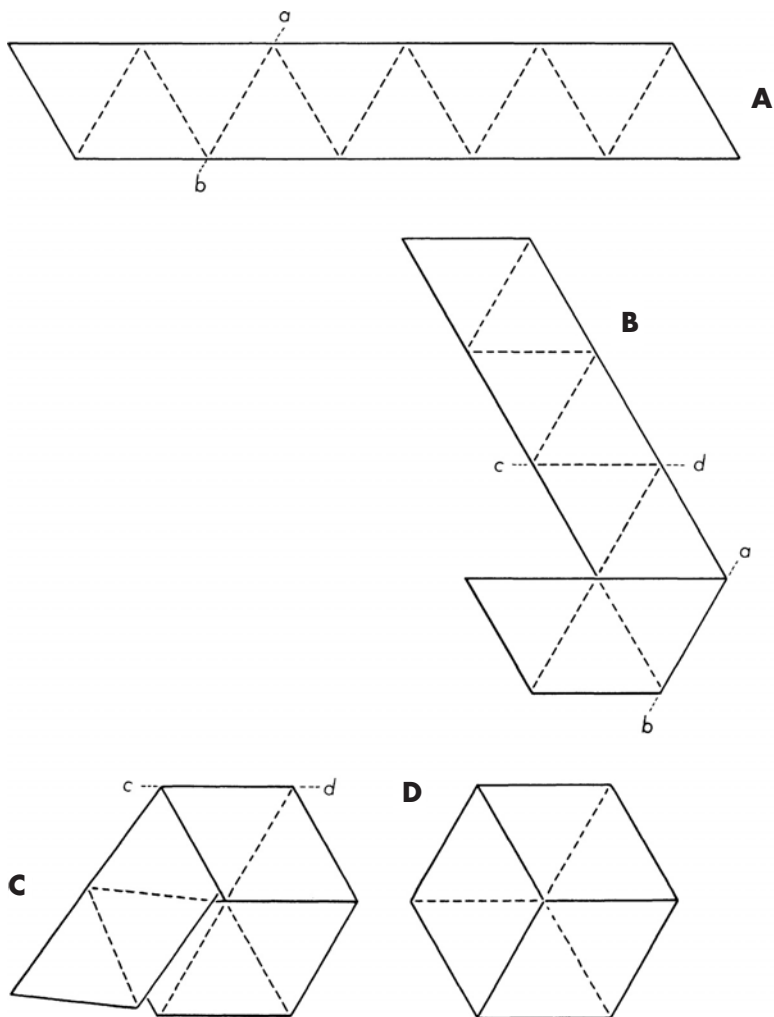**Figure 1.** Trihexaflexagon is constructed by cutting a strip of paper so that it may be marked off in 10 equilateral triangles (A). The strip is folded backward along the line *ab* and turned over (B). It is then folded backward again along the line *cd* and the next-to-the-last-triangle placed on top of the first (C). The last triangle is now folded backward and glued to the other side of the first (D). Fairly stiff paper at least an inch and a half wide is recommended. (Artist: Bunji Tagawa)
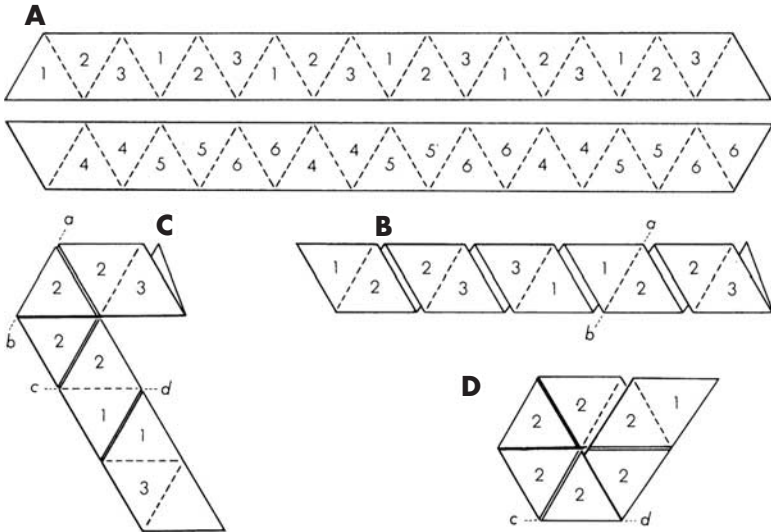
Figure 2. Hexahexaflexagon is constructed by cutting a strip of paper so that it may be marked off in 19 triangles (A). The triangles on one side are numbered 1, 2, and 3; the triangles on the other, 4, 5, and 6. A similar pattern of colors or geometrical figures may also be used. The strip of paper is then folded as shown (B–D). It can be flexed to show six different faces. (Artist: Bunji Tagawa)

Soon "flexagons" were appearing in profusion at the lunch and dinner tables. The Flexagon Committee was organized to probe further into the mysteries of flexigation. The other committee members besides Stone were Bryant Tuckerman, a graduate student of mathematics; Richard P. Feynman, a graduate student in physics; and John W. Tukey, a young mathematics instructor.

The models were named hexaflexagons – *hexa* for their hexagonal form and *flexagon* for their ability to flex. Stone's first model is a trihexaflexagon (*tri* for the three different faces that can be brought into view); his elegant second structure is a hexahexaflexagon (for its six faces).

To make a hexahexaflexagon, you start with a strip of paper (the tape used in adding machines serves admirably), which is divided into 19 equilateral triangles (see Figure 2A). You number the triangles on one side of the strip 1, 2, and 3, leaving the nineteenth triangle blank, as shown in the drawing. On the opposite side the triangles are numbered 4, 5, and 6, according to the scheme shown.

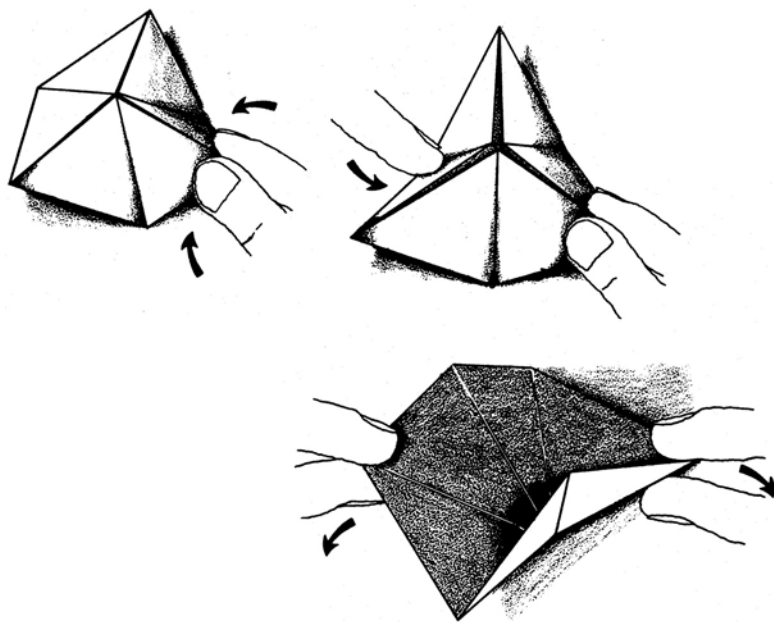**Figure 3.** Trihexaflexagon is flexed by pinching together two of its triangles (*top*). The inner edge of the two opposite triangles may be opened with the other hand (*bottom*). If the figure cannot be opened, the adjacent pair of triangles is pinched. If the figure opens, it can be turned inside out, revealing a side that was not visible before. (Artist: Bunji Tagawa)

Now you fold the strip so that the same underside numbers face each other – 4 on 4, 5 on 5, 6 on 6, and so on. The resulting folded strip, illustrated in Figure 2B, is then folded back on the lines *ab* and *cd* (Figure 2C), forming the hexagon (Figure 2D); finally the blank triangle is turned under and pasted to the corresponding blank triangle on the other side of the strip. All this is easier to carry out with a marked strip of paper than it is to describe.

If you have made the folds properly, the triangles on one visible face of the hexagon will all be numbered 1, and on the other face they will all be numbered 2. Your hexahexaflexagon is now ready for flexing. Pinch two adjacent triangles together (see Figure 3), bending the paper along the line between them, and push in the opposite corner; the figure may then open up to face 3 or 5. By random flexing you should be able to find the other faces without much difficulty. Faces 4, 5, and 6 are a bit harder to uncover than faces 1, 2, and 3. At
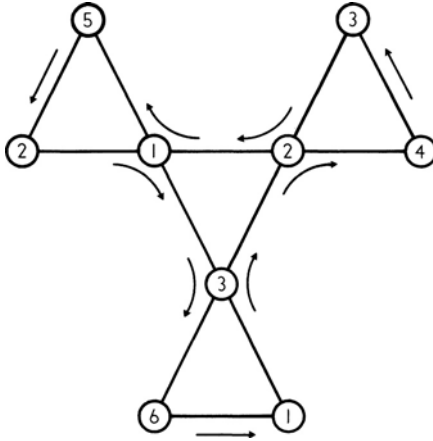
Figure 4. Diagram of a Tuckerman traverse on a hexahexaflexagon.

times you may find yourself trapped in an annoying cycle that keeps returning the same three faces over and over again.

Tuckerman quickly discovered that the simplest way to bring out all the faces of any flexagon was to keep flexing it at the same corner until it refused to open, then to shift to an adjacent corner. This procedure, known as the Tuckerman traverse, will bring up the six faces of a hexahexa in a cycle of 12 flexes, but faces 1, 2, and 3 turn up three times as often as faces 4, 5, and 6. A convenient way to diagram a Tuckerman traverse is shown in Figure 4, the arrows indicating the order in which the faces are brought into view. This type of diagram can be applied usefully to the traversing of any type of flexagon. When the model is turned over, a Tuckerman traverse runs the same cycle in reverse order.

By lengthening the chain of triangles, the committee discovered, one can make flexagons with 9, 12, 15 or more faces: Tuckerman managed to make a workable model with 48! He also found that with a strip of paper cut in a zigzag pattern (i.e., a strip with sawtooth rather than straight edges) it is possible to produce a tetrahexaflexagon (four faces) or a pentahexaflexagon. There are three different hexahexaflexagons – one folded from a straight strip, one from a chain bent into a hexagon, and one from a form that somewhat resembles a three-leaf clover. The decahexaflexagon (10 faces) has 82 different variations, all folded from weirdly bent strips. Flexagons can be formed with any desired number of faces, but beyond 10 the

number of different species for each increases at an alarming rate. All even-numbered flexagons, by the way, are made of strips with two distinct sides, but those with an odd number of faces have only a single side, like a Moebius surface.

A complete mathematical theory of flexigation was worked out in 1940 by Tukey and Feynman. It shows, among other things, exactly how to construct a flexagon of any desired size or species. The theory has never been published, though portions of it have since been rediscovered by other mathematicians. Among the flexigators is Tuckerman's father, the distinguished physicist Louis B. Tuckerman, who was formerly with the National Bureau of Standards. Tuckerman senior devised a simple but efficient tree diagram for the theory.

The Japanese attack on Pearl Harbor called a halt to the committee's flexigation program, and war work soon scattered the four charter members to the winds. Stone became a lecturer in mathematics at the University of Manchester in England and is now at the University of Rochester. Feynman was a famous theoretical physicist at the California Institute of Technology. Tukey, a professor of mathematics at Princeton, has made brilliant contributions to topology and to statistical theory, which have brought him worldwide recognition. Tuckerman is a mathematician at IBM's research center in Yorktown Heights, New York.

One of these days, the committee hopes to reconvene and write a paper or two, which will be the definitive exposition of flexagon theory. Until then, the rest of us are free to flex our flexagons and see how much of the theory we can discover for ourselves.

ADDENDUM

In constructing flexagons from paper strips, it is a good plan to crease all the fold lines back and forth before folding the model. As a result, the flexagon flexes much more efficiently. Some readers have made more durable models by cutting triangles from poster board or metal and joining them with small pieces of tape or gluing them to one long piece of tape, leaving spaces between them for flexing. Louis Tuckerman keeps on hand a steel strip of such size that by
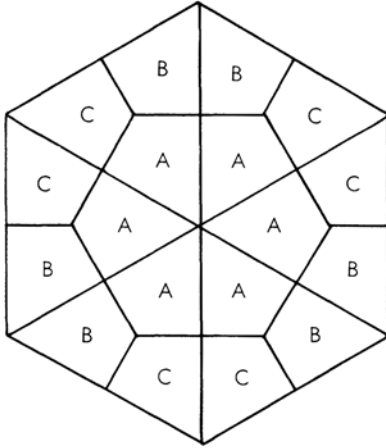
Figure 5. (Artist: Bunji Tagawa.)

wrapping paper tape of a certain width around it he can quickly pro-
duce a folded strip of the type shown in Figure 2. This saves consid-
erable time in making flexagons from straight chains of triangles.

Readers have sent to me a large variety of ways in which flexagon
faces could be decorated to make interesting puzzles or display
striking visual effects. Each face of the hexahexa, for example,
appears in at least two different forms, owing to a rotation of the
component triangles relative to each other. Thus if we divide each
face as shown in Figure 5, using different colors for the A, B, and C
sections, the same face may appear with the A sections in the center
as shown, or with the B or C sections in the center. Figure 6 shows
how a geometrical pattern may be drawn on one face so as to appear
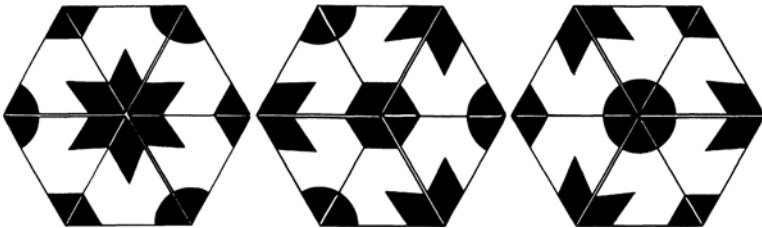in three different configurations.

Figure 6. (Artist: Bunji Tagawa.)

Of the 18 possible faces that can result from a rotation of the tri-
angles, three are impossible to achieve with a hexahexa of the type
made from a straight strip. This suggested to one correspondent
the plan of pasting parts of three different pictures on each face
so that by flexing the model properly, each picture could presum-
ably be brought together at the center while the other two would
be fragmented around the rim. On the three inner hexagons that
cannot be brought together, he pasted the parts of three pictures of
comely, undraped young ladies to make what he called a hexahexa-
frustragon. Another reader wrote that he achieved similar results by
pasting together two adjacent triangular faces. This prevents one
entire face from flexing into view, although the victim can see that it
exists by peeking into the model's interior.

The statement that only 15 different patterns are possible on the
straight-strip hexahexa must be qualified. An unsymmetrical color-
ing of the faces discloses the curious fact that three of these fifteen
patterns have mirror-image partners. If you number the inner cor-
ners of each pattern with digits from 1 to 6, writing them in clock-
wise order, you will find that three of these patterns turn up with the
same digits in counterclockwise order. Bearing this asymmetry in
mind, one can say that the six faces of this hexahexa exhibit a total of
18 different configurations. This was first called to my attention by
Albert Nicholas, professor of education at Monmouth College, Mon-
mouth, Illinois, where the making of flexagons became something
of a craze in the early months of 1957.

I do not know who first used a printed flexagon as an advertis-
ing premium or greeting card. The earliest sent to me was a tri-
hexa distributed by the Rust Engineering Company of Pittsburgh to
advertise their service award banquet in 1955. A handsome hexa-
hexa, designed to display a variety of colored snowflake patterns,
was used by *Scientific American* for its 1956 Christmas card.

For readers who would like to construct and analyze flexagons
other than the two described in this chapter, here is a quick run-
down on some low-order varieties.

1.   *The unahexa.* A strip of three triangles can be folded flat and
      the opposite ends joined to make a Moebius strip with a tri-
      angular edge. (For a more elegant model of a Moebius band

with triangular edge, see Chapter 7.) Since it has one side
only, made up of six triangles, one might call it a unahexa-
flexagon, though of course it isn't six-sided and it doesn't flex.

2. *The duahexa.* Simply a hexagon cut from a sheet of paper. It
   has two faces but doesn't flex.
3. *The trihexa.* This has only the one form described in this
   chapter.
4. *The tetrahexa.* This flexagon likewise has only one form. It is
   folded from the crooked strip shown in Figure 7A.
5. *The pentahexa.* One form only. Folded from the strip in Fig-
   ure 7B.
6. *The hexahexa.* There are three varieties, each with unique
   properties. One of them is described in this chapter. The
   other two are folded from the strips pictured in Figure 7C.
7. *The heptahexa.* This flexagon can be folded from the three
   strips shown in Figure 7D. The first strip can be folded in two
   different ways, making four varieties in all. The third form,
   folded from the overlapping figure-8 strip, is the first of what
   Louis Tuckerman calls the "street flexagons." Its faces can
   be numbered so that a Tuckerman traverse will bring upper-
   most the seven faces in serial order, like passing the street
   numbers on a row of houses.

The octahexa has 12 distinct varieties; the enneahexa has 27; and
the decahexa, 82. The exact number of varieties of each order can
be figured in more than one way, depending on how you define
a distinct variety. For example, all flexagons have an asymmetric
structure that can be right-handed or left-handed, but mirror-image
forms should hardly be classified as different varieties. For details on
the number of nonequivalent hexaflexagons of each order, consult
the paper by Oakley and Wisner listed in the bibliography.

Straight chains of triangles produce only hexaflexagons with
orders that are multiples of three. One variety of a 12-faced hexa
is particularly easy to fold. Start with a straight chain twice as long
as the one used for the hexahexa. "Roll" it into the form shown in
Figure 2B. The strip is now the same length as the one used for the
hexahexa. Fold this rolled strip exactly as if you were making a hex-
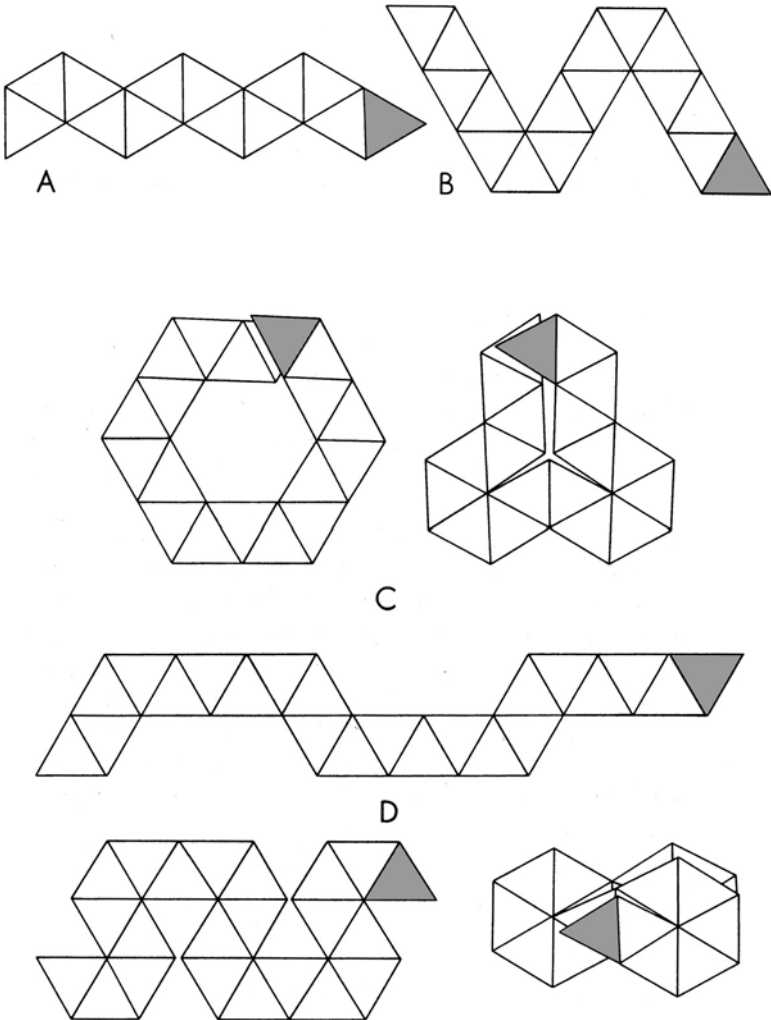ahexa. The result is a dodecahexaflexagon.

Figure 7. Crooked strips for folding hexaflexagons. The shaded triangles are tabs for pasting. (Artist: Bunji Tagawa)

In experimenting with higher-order flexagons, a handy rule to bear in mind is that the sum of the number of leaves (thicknesses of paper) in two adjacent triangular sections always equals the number of faces. It is interesting to note also that if each face of a flexagon is given a number or symbol, and the symbol marked on each