CAMBRIDGE STUDIES IN
ADVANCED MATHEMATICS

Lectures in Logic and Set Theory Volume 2

This two-volume work bridges the gap between introductory expositions of logic or set theory on one hand, and the research literature on the other. It can be used as a text in an advanced undergraduate or beginning graduate course in mathematics, computer science, or philosophy. The volumes are written in a user-friendly conversational lecture style that makes them equally effective for self-study or class use.

Volume 2, on formal (ZFC) set theory, incorporates a self-contained "Chapter 0" on proof techniques so that it is based on formal logic, in the style of Bourbaki. The emphasis on basic techniques will provide the reader with a solid foundation in set theory and sets a context for the presentation of advanced topics such as absoluteness, relative consistency results, two expositions of Gödel's constructible universe, numerous ways of viewing recursion, and a chapter on Cohen forcing.

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Volume 2: Set Theory

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To the memory of my parents

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Preface

This volume contains the basics of Zermelo-Fraenkel axiomatic set theory. It is situated between two opposite poles: On one hand there are elementary texts that familiarize the reader with the vocabulary of set theory and build set-theoretic tools for use in courses in analysis, topology, or algebra – but do not get into metamathematical issues. On the other hand are those texts that explore issues of current research interest, developing and applying tools (constructibility, absoluteness, forcing, etc.) that are aimed to analyze the inability of the axioms to settle certain set-theoretic questions.

Much of this volume just "does set theory", thoroughly developing the theory of ordinals and cardinals along with their arithmetic, incorporating a careful discussion of diagonalization and a thorough exposition of induction and inductive (recursive) definitions. Thus it serves well those who simply want tools to apply to other branches of mathematics or mathematical sciences in general (e.g., theoretical computer science), but also want to find out about some of the subtler results of modern set theory.

Moreover, a fair amount is included towards preparing the advanced reader to read the research literature. For example, we pay two visits to Gödel's constructible universe, the second of which concludes with a proof of the relative consistency of the axiom of choice and of the generalized continuum hypothesis with ZF. As such a program requires, I also include a thorough discussion of formal interpretations and absoluteness. The lectures conclude with a short but detailed study of Cohen forcing and a proof of the non-provability in ZF of the continuum hypothesis.

The level of exposition is designed to fit a spectrum of mathematical sophistication, from third-year undergraduate to junior graduate level (each group will find here its favourite chapters or sections that serve its interests and level of preparation).

The volume is self-contained. Whatever tools one needs from mathematical logic have been included in Chapter I. Thus, a reader equipped with a combination of sufficient mathematical maturity and patience should be able to read it and understand it. There is a trade-off: the less the maturity at hand, the more the supply of patience must be. To pinpoint this "maturity": At least two courses from among calculus, linear algebra, and discrete mathematics at the junior level should have exposed the reader to sufficient diversity of mathematical issues and proof culture to enable him or her to proceed with reasonable ease.

A word on approach. I use the Zermelo-Fraenkel axiom system *with* the axiom of choice (AC). This is the system known as ZFC. As many other authors do, I simplify nomenclature by allowing "proper classes" in our discussions as part of our metalanguage, but not in the formal language.

I said earlier that this volume contains the "basics". I mean this characterisation in two ways: One, that all the fundamental tools of set theory as needed elsewhere in the mathematical sciences are included in detailed exposition. Two, that I do not present any applications of set theory to other parts of mathematics, because space considerations, along with a decision to include certain advanced relative consistency results, have prohibited this.

"Basics" also entails that I do not attempt to bring the reader up to speed with respect to current research issues. However, a reader who has mastered the advanced metamathematical tools contained here will be able to read the literature on such issues.

The title of the book reflects two things: One, that all good *short* titles are taken. Two, more importantly, it advertises my conscious effort to present the material in a conversational, user-friendly lecture style. I deliberately employ classroom mannerisms (such as "pauses" and parenthetical "why"s, "what if"s, and attention-grabbing devices for passages that I feel are important). This aims at creating a friendly atmosphere for the reader, especially one who has decided to study the topic without the guidance of an instructor. Friendliness also means steering clear of the terse axiom-definition-theorem recipe, and explaining how some concepts were *arrived at* in their present form. In other words, what makes things tick. Thus, I approach the development of the key concepts of ordinals and cardinals, *initially* and *tentatively*, in the manner they were originally introduced by Georg Cantor (paradox-laden and all). Not only does this afford the reader an understanding of why the modern (von Neumann) approach is superior (and contradiction-free), but it also shows what it tries to accomplish. In the same vein, Russell's paradox is visited no less than three times, leaving us in the end with a firm understanding that it has nothing to do with the "truth" or otherwise of the much-maligned statement "$x \in x$" but it is just the result of a *diagonalization* of the type Cantor originally taught us.

A word on coverage. Chapter I is our "Chapter 0". It contains the tools needed to enable us do our job properly – a bit of mathematical logic, certainly no more than necessary. Chapter II informally outlines what we are about to describe axiomatically: the universe of all the "real" sets and other "objects" of our intuition, a caricature of the von Neumann "universe". It is explained that the whole fuss about axiomatic set theory[†] is to have a *formal* theory derive true statements about the von Neumann sets, thus enabling us to get to *know* the nature and structure of this universe. If this is to succeed, the chosen axioms must be seen to be "true" in the universe we are describing.

To this end I ensure via *informal discussions* that every axiom that is introduced is seen to "follow" from the principle of the formation of sets by stages, or from some similarly plausible principle devised to keep paradoxes away. In this manner the reader is constantly made aware that we are building a *meaningful* set theory that has relevance to mathematical intuition and expectations (the "real" mathematics), and is not just an artificial choice of a contradiction-free set of axioms followed by the mechanical derivation of a few theorems.

With this in mind, I even make a case for the plausibility of the axiom of choice, based on a popularization of Gödel's constructible universe argument. This occurs in Chapter IV and is informal.

The set theory we do allows *atoms* (or *Urelemente*),[‡] just like Zermelo's. The re-emergence of atoms has been defended aptly by Jon Barwise (1975) and others on technical merit, especially when one does "restricted set theories" (e.g., theory of admissible sets).

Our own motivation is not technical; rather it is philosophical and pedagogical. We find it extremely counterintuitive, especially when addressing undergraduate audiences, to tell them that all their familiar mathematical objects – the "stuff of mathematics" in Barwise's words – *are* just perverse "box-in-a-box-in-a-box . . . " formations built from an infinite supply of empty boxes. For example, should I be telling my undergraduate students that their *familiar* number "2" *really is* just a short name for something like "$\boxed{\square\boxed{\square}}$" ? And what will I tell them about "$\sqrt{2}$"?

[†] O.K., maybe not the *whole* fuss. Axiomatics also allow us to meaningfully ask, and attempt to answer, metamathematical questions of derivability, consistency, relative consistency, independence. But in this volume much of the fuss is indeed about learning set theory.

[‡] Allows, but does not insist that there are any.

Some mathematicians have said that set theory (without atoms) speaks only of *sets* and it chooses *not* to speak about objects such as cows or fish (colourful terms for urelements). Well, it does too! Such ("atomless") set theory is known to be perfectly capable of *constructing* "artificial" cows and fish, and can then proceed to talk about such animals as much as it pleases.

While atomless ZFC has the ability to construct or codify all the familiar mathematical objects in it, it does this so well that it betrays the prime directive of the axiomatic method, which is to have a theory that *applies* to diverse concrete (*meta* – i.e., outside the theory and in the realm of "everyday math") mathematical systems. Group theory and projective geometry, for example, fulfill the directive.

In atomless ZFC the opposite appears to be happening: One is asked to *embed* the known mathematics into the formal system.

We prefer a set theory that allows both artificial and real cows and fish, so that when we want to illustrate a point in an example utilizing, say, the everyday set of integers, $\mathbb{Z}$, we can say things like "let the atoms (be interpreted to) include the members of $\mathbb{Z}$ . . . ".

But how about technical convenience? Is it not hard to include atoms in a formal set theory? In fact, not at all!

A word on exposition devices. I freely use a pedagogical feature that, I believe, originated in Bourbaki's books – that is, marking an important or difficult topic by placing a "winding road" sign in the margin next to it. I am using here the same symbol that Knuth employed in his TEXbook, namely, ☡, marking with it the beginning and end of an important passage.

Topics that are advanced, or of the "read at your own risk" type, *can be omitted without loss of continuity*. They are delimited by a double sign, ☡☡.

Most chapters end with several exercises. I have stopped making attempts to sort exercises between "hard" and "just about right", as such classifications are rather subjective. In the end, I'll pass on to you the advice one of my professors at the University of Toronto used to offer: "Attempt all the problems. Those you can do, don't do. Do the ones you cannot".

What to read. Just as in the advice above, I suggest that you read everything that you do not already know if time is no object. In a class environment the coverage will depend on class length and level, and I defer to the preferences of the instructor. I suppose that a fourth-year undergraduate audience ought to see the informal construction of the constructible universe in Chapter IV, whereas a graduate audience would rather want to see the formal version in Chapter VI. The latter group will probably also want to be exposed to Cohen forcing.

Acknowledgments. I wish to thank all those who taught me, a group that is too large to enumerate, in which I must acknowledge the presence and influence of my parents, my students, and the writings of Shoenfield (in particular, 1967, 1978, 1971).

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I finally wish to thank Donald Knuth and Leslie Lamport for their typesetting systems TeX and LaTeX that make technical writing fun (and also empower authors to load the pages with ☡ and other signs).

George Tourlakis
Toronto, March 2002