**Smooth Compactifications of Locally Symmetric Varieties**

Second Edition

The new edition of this celebrated and long-unavailable book preserves much of the content and structure of the original, which is still unrivalled in its presentation of a universal method for the resolution of a class of singularities in algebraic geometry. At the same time, the book has been completely re-typeset, errors have been eliminated, proofs have been streamlined, the notation has been made consistent and uniform, and an index and a guide to recent literature have been added.

The authors begin by reviewing, in Chapter I, key results in the theory of toroidal embeddings and by explaining examples that illustrate the theory. Chapter II develops the theory of open self-adjoint homogeneous cones and their polyhedral reduction theory. Chapter III is devoted to basic facts on hermitian symmetric domains and culminates in the construction of toroidal compactifications of their quotients by an arithmetic group. The final chapter considers several applications of the general results.

The book brings together ideas from algebraic geometry, differential geometry, representation theory and number theory, and will continue to prove of value for researchers and graduate students in these areas.

# Smooth Compactifications of Locally Symmetric Varieties

Second Edition

AVNER ASH
*Boston College*

DAVID MUMFORD
*Brown University*

MICHAEL RAPOPORT
*University of Bonn*

YUNG-SHENG TAI
*Haverford College*

With the collaboration of

PETER SCHOLZE
*University of Bonn*

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521739559

First edition published in 1975 by Math Sci Press


Second edition published in 2010 by Cambridge University Press


Printed in the United Kingdom at the University Press, Cambridge

*A catalogue record for this publication is available from the British Library*

ISBN 978-0-521-73955-9 paperback

# Contents

# Preface to the second edition

When CUP approached us with the proposal of a second edition to our book, we first consulted graduate students and younger colleagues to test this idea on them. Their enthusiastic response convinced us of the soundness of the proposition.

In order to keep this project within realistic bounds, we did not rewrite the book, but rather TEX-ed the original text and corrected mistakes that have come to our attention. We also smoothed somewhat the presentation and homogenized the notation. Finally, in order to increase its usability, we added an index.

So, all in all, this is still essentially the same book. In particular, the text of this new edition does not reflect the developments in the field in the last 30 years. To compensate for this, we added a guide to the more recent literature at the end of the book.

In this whole project we were assisted by Peter Scholze, who read the whole manuscript, corrected many mistakes, and helped us with the proof-reading. We thank him heartily. We also thank Y. May, who assisted us in TEX-problems.

We also thank all those who pointed out mistakes in the first edition and often indicated to us how to correct them: we are thus grateful to C.-L. Chai, E. Looijenga, R. Pink, Y. Namikawa, and I. Satake. Finally, we thank the staff of CUP, and particularly D. Tranah, for their expert cooperation.

Avner Ash,
David Mumford,
Michael Rapoport,
Yung-sheng Tai.

# Preface to the first edition

Let $D$ be a bounded symmetric domain and let $\Gamma$ be a neat (see Ch. III, §7) arithmetic subgroup of $\mathrm{Aut}\,(D)^o$. The goal of this monograph is the construction of a family of non-singular† compactifications $\overline{D/\Gamma}$ of $D/\Gamma$. This theory was announced and described in rough outline in [2]. Very similar ideas were developed independently by Satake in [3]. Both of us were following the path indicated by the work of Igusa when $\Gamma = \mathrm{Sp}\,(2n, \mathbb{Z})$ and by Hirzebruch when $\Gamma = \mathrm{SL}(2, \mathscr{O})$, where $\mathscr{O}$ = integers in a real quadratic field.

Here is an outline of the monograph. Since this work builds heavily on [1] (referred to as TE I below), we review quickly some of these results and add some comments particular to the complex case in Ch. I, §§1–3. Then, in Ch. I, §§4,5, we describe smooth compactifications of two surfaces $D/\Gamma$, in order to illustrate the general theory which follows (actually, in §4, $D$ is not bounded – it is $\Delta \times \mathbb{C}$ – so this is not strictly a special case). Chapter II, by A. Ash, is devoted to self-adjoint homogeneous cones. The main result is a comparison of Siegel sets and polyhedral subcones inside such homogeneous cones. These results are essential for the construction of $\overline{D/\Gamma}$. The construction itself is taken up in Ch. III. The final results require considerable notation to state, but may be found in Ch. III, §§5, 7. The principal technical contribution here is M. Rapoport's calculation of the Satake topology on $D^* = D \bigcup (\text{rat. boundary comp.})$ in terms of the presentation of $D$ as a Siegel domain of third kind, which is crucial to proving that our $\overline{D/\Gamma}$ is Hausdorff. The final chapter by Y. S. Tai adds two important results. Firstly, he applies the construction to prove that $D/\Gamma$ is a variety "of general type" in Kodaira's classification when $\Gamma$ is small enough. Secondly, although our general $\overline{D/\Gamma}$ is only an analytic compactification of $D/\Gamma$, he shows that many of these $\overline{D/\Gamma}$'s are indeed projective varieties.

One of the main obstacles in our research was that none of us were symmetric space specialists when we began, and, of course, roots are the name of the game throughout. For our sake as well as the reader's, we thought it useful to include a considerable amount of expository material in the hope of making

† We are mainly concerned with a larger class of compactifications with "toroidal" singularities on the boundary. Within this class, there are plenty of non-singular compactifications, but these do not play any special role in our study.

the monograph more self-contained. We were greatly aided by similar expository projects of P. Deligne and I. Satake, who graciously lent us their notes. In general, the expository sections emphasize the geometric aspects somewhat more than the references, and, in particular, develop the ideas in the form in which we need them. Experts can skim rapidly through Ch. II, §§1–3 (note, however, the very crucial tie-up between Pierce decompositions and split tori which appears to be new) and Ch. III, §§2–4 (note here the key role played by the intermediate open set $D(F): D \subset D(F) \subset \check{D}$, $D(F) = U(F)_{\mathbb{C}} \cdot D$, in the construction of the Siegel Domain realization).

I hope that the space $\overline{D/\Gamma}$ here constructed will have other applications in the theory of automorphic forms, e.g., to calculating invariants of the field $\mathbb{C}(D/\Gamma)$ and the dimension of the spaces of automorphic forms. Besides these applications, the theory can hopefully be pushed further in three essential directions: (i) at least for $D = \mathrm{Sp}(2n,\mathbb{R})/K$, extend it to a construction of a scheme $\overline{D/\Gamma}$ over $\mathbb{Z}$; (ii) to extend Hirzebruch's proportionality theorems to the non-compact case; (iii) in view of the fact that the results describe concretely the degeneration of Hodge structures of a very special type – find an extension of them, combining the ideas of Ch. III, §7 with Schmid's results on families of Hodge structures over $\mathring{\Delta}$, to describe asymptotically all families of Hodge structures on $(\mathring{\Delta})^k$.

David Mumford

## Authorship of the various chapters

Chapter I: David Mumford

Chapter II: Avner Ash

Chapter III: Michael Rapoport and David Mumford

Chapter IV: Yung-sheng Tai

## References

[1] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, *Toroidal Embeddings I,* Lecture Notes in Mathematics 339. Berlin: Springer 1972.†

[2] D. Mumford, A new approach to compactifying locally symmetric varieties, in *Proceedings of the Tata Institute Colloquium, Jan. 1973*, Oxford Univ. Press, 1975.

[3] I. Satake, On the arithmetic of tube domains, *Bull. Amer. Math. Soc.*, **79** (1973), 1076-1094.

† This is referred as 'TE I' throughout the monograph.