PART 0

General Remarks and Basic Concepts

1 The Classical Field Equations

It is a remarkable aspect of the "unreasonable effectiveness of mathematics in the natural sciences" (Wigner 1960) that a handful of equations are sufficient to describe mathematically a vast number of physically disparate phenomena, at least at some level of approximation. Key reasons are the isotropy and uniformity of space-time (at least locally), the attendant conservation laws,[1] and the useful range of applicability of linear approximations to constitutive relations.

After a very much abbreviated survey of the principal properties of vector fields, we present a summary of these fundamental equations and associated boundary conditions, and then describe several physical contexts in which they arise. The initial chapters of a book on any specific discipline give a far better derivation of the governing equations for that discipline than space constraints permit here. Our purpose is, firstly, to remind the reader of the meaning accorded to the various symbols in any specific application and of the physics that they describe and, secondly, to show the similarity among different phenomena.

The final section of this chapter is a very simple-minded description of the method of eigenfunction expansion systematically used in many of the applications treated in this book. The starting point is an analogy with vectors and matrices in finite-dimensional spaces and the approach is purposely very elementary; a "real" theory is to be found in Part III of the book.

1.1 Vector fields

Throughout most of the book the symbol u is used to denote the unknown function. Although mathematically unnecessary, a distinction is made between the time variable t and the space variables denoted, in Cartesian coordinates, by $\mathbf{x} = (x, y, z)$; vectors are shown in bold face.

The gradient $\boldsymbol{\nabla} u$ of a scalar function u is a vector with Cartesian components

$$\boldsymbol{\nabla} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right) = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}, \tag{1.1.1}$$

[1] Noether's theorem asserts that any differentiable symmetry of the action (i.e., the integral of the Lagrangian) of a physical system has a corresponding conservation law (see e.g. Lanczos 1970, p. 401; Goldstein 1980, p. 588; Weinberg 1995, vol. 1, p. 307; Srednicki 2007, p. 133).

in which **i**, **j** and **k** are the unit vectors in the x-, y- and z-directions respectively. Physically, $(\boldsymbol{\nabla} u)\cdot \mathrm{d}\mathbf{x}$ represents the increment of u when its spatial argument is changed from $\mathbf{x} = (x,\ y,\ z)$ to $\mathbf{x} + \mathrm{d}\mathbf{x} = (x + \mathrm{d}x\,,\ y + \mathrm{d}y\,,\ z + \mathrm{d}z\,)$. The same meaning is attached to $(\boldsymbol{\nabla} u)\cdot \mathrm{d}\mathbf{x}$ in any other coordinate system (ξ, η, ζ) and, for this reason, the component of the gradient in the ξ-direction, for example, is not necessarily given by $\partial u/\partial \xi$;[2] the form of the gradient in cylindrical and spherical coordinate systems is given in Tables 6.4 p. 148 and 7.3 p. 173, respectively.

The divergence of a vector **A** with Cartesian components $(A_x,\ A_y,\ A_z)$ is

$$\boldsymbol{\nabla}\cdot\mathbf{A} \;=\; \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{1.1.2}$$

The physical meaning of $\boldsymbol{\nabla}\cdot\mathbf{A}$ is made evident by the divergence theorem (pp. 400 and 592) which implies that

$$(\boldsymbol{\nabla}\cdot\mathbf{A})(\mathbf{x}) \;=\; \lim_{\Delta V\to 0}\frac{1}{\Delta V}\oint_S \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}S\,, \tag{1.1.3}$$

in which S is the boundary of a small volume ΔV centered at **x**, **n** is the outwardly directed unit normal and the limit is understood in the sense that all dimensions of ΔV tend to zero approximately at the same rate. Thus $(\Delta V)\,\boldsymbol{\nabla}\cdot\mathbf{A}$ represents the net flux of **A** out of the elementary volume ΔV.

A third important differential vector operator is the curl:

$$\boldsymbol{\nabla}\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k}. \tag{1.1.4}$$

Similarly to (1.1.3), we can get some insight into the physical meaning of this quantity from Stokes's theorem (p. 401) which permits us to write

$$\mathbf{n}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) \;=\; \lim_{\Delta S\to 0}\frac{1}{\Delta S}\oint_L \mathbf{A}\cdot\mathbf{t}\,\mathrm{d}\ell\,. \tag{1.1.5}$$

Here L is a small planar loop enclosing an area ΔS which has a unit normal **n**; **t** is the unit vector tangent to L oriented in the direction of the fingers of the right hand when the thumb points along **n**; $(\Delta S)\mathbf{n}\cdot(\boldsymbol{\nabla}\times\mathbf{A})$ represents therefore the circulation of **A** around the small loop encircling the area ΔS.

Just as in the case of the gradient, although the physical meaning associated with divergence and curl is the same in all coordinate systems, the forms shown in (1.1.2) and (1.1.4) are only valid in Cartesian coordinates because this is the only system in which the unit vectors along the coordinate lines are constant. In a general orthogonal system $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ a vector field is represented as

$$\mathbf{A} \;=\; A_1(\mathbf{x})\mathbf{e}_1 + A_2(\mathbf{x})\mathbf{e}_2 + A_3(\mathbf{x})\mathbf{e}_3 \;=\; \sum_{k=1}^{3} A_k\mathbf{e}_k. \tag{1.1.6}$$

[2] $\partial u/\partial\xi$ is the component of the gradient in the ξ-direction only when $\mathrm{d}\xi$ represents the actual distance between the points having coordinates (ξ, η, ζ) and $(\xi + \mathrm{d}\xi\,, \eta, \zeta)$. Evidently, this cannot be the case, for example, if ξ is an angular coordinate.

Then, using the standard rules of vector calculus to express $\boldsymbol{\nabla}\cdot[f\,\mathbf{V}]$ and $\boldsymbol{\nabla}\times[f\,\mathbf{V}]$, we would have

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \sum_{k=1}^{3}[(\boldsymbol{\nabla}A_k)\cdot\mathbf{e}_k + A_k\boldsymbol{\nabla}\cdot\mathbf{e}_k], \qquad \boldsymbol{\nabla}\times\mathbf{A} = \sum_{k=1}^{3}[(\boldsymbol{\nabla}A_k)\times\mathbf{e}_k + A_k\boldsymbol{\nabla}\times\mathbf{e}_k]. \tag{1.1.7}$$

Expressions for the divergence and the curl of a vector field in cylindrical and spherical coordinate systems are shown in Tables 6.4 p. 148 and 7.3 p. 173, respectively.

In the following a central role is played by the Laplace operator, or *Laplacian*,

$$\nabla^2 u = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \tag{1.1.8}$$

While the form $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}u)$ is valid in all coordinate systems, it follows from the previous considerations that the last form is only valid in Cartesian coordinates; the form of this operator in cylindrical and spherical coordinate systems is shown in Tables 6.4 p. 148 and 7.3 p. 173, respectively. For a vector field $\mathbf{A}$, in Cartesian coordinates we have

$$\nabla^2\mathbf{A} = (\nabla^2 A_x)\mathbf{i} + (\nabla^2 A_y)\mathbf{j} + (\nabla^2 A_z)\mathbf{k}, \tag{1.1.9}$$

but in a general coordinate system the Laplacian of a vector is defined using the relation

$$\nabla^2\mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A}, \tag{1.1.10}$$

which is an identity in Cartesian coordinates and provides a consistent definition of $\nabla^2\mathbf{A}$ in non-Cartesian coordinate systems.

Two standard vector identities are

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}u) = 0, \qquad \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0. \tag{1.1.11}$$

Furthermore, the following two implications are well known:

$$\boldsymbol{\nabla}\times\mathbf{C} = 0 \iff \mathbf{C} = \boldsymbol{\nabla}u, \tag{1.1.12}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \iff \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{1.1.13}$$

In (1.1.12) u is the *scalar potential*[3] of the field $\mathbf{C}$ and, in (1.1.13), $\mathbf{A}$ is the *vector potential* of the field $\mathbf{B}$. Since the equation $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ implies that only two of the three scalars that constitute the field $\mathbf{B}$ are independent, the same must be true of $\mathbf{A}$. Thus, (1.1.13) must be insufficient to uniquely specify the vector potential and, indeed, there is the freedom to add various subsidiary conditions to make $\mathbf{A}$ unique without affecting the physical field $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. These conditions, examples of which will be found in §1.5, are known as *gauge conditions*.

[3] The relation (1.1.12) holds in a simply connected domain. In a multiply connected domain the local condition $\boldsymbol{\nabla}\times\mathbf{C} = 0$ must be supplemented by the global requirement that the circulation of $\mathbf{C}$ around any closed path vanish.

For a general vector field $\mathbf{V}$ we have the *Helmholtz* or *Hodge decomposition* (see e.g. Morse & Feshbach 1953, p. 53)

$$\mathbf{V} = \boldsymbol{\nabla} u + \nabla \times \mathbf{A}, \tag{1.1.14}$$

with $\nabla^2 u = \boldsymbol{\nabla} \cdot \mathbf{V}$. The two terms, $\boldsymbol{\nabla} u$ and $\nabla \times \mathbf{A}$, are the *longitudinal* and *transverse* parts of $\mathbf{V}$.

The addition of the gradient of a scalar function to the vector potential $\mathbf{A}$ does not affect the fields $\mathbf{B}$ or $\mathbf{V}$ and, because of this freedom, in some situations it is possible to assume without loss of generality particularly convenient forms for the vector potential. For example, if the problem is planar, one may take $\mathbf{A} = \boldsymbol{\nabla} \times (\psi \mathbf{k})$ in which $\mathbf{k}$ is a constant vector orthogonal to the plane and ψ a scalar function. With axial symmetry, one may take $\mathbf{A} = \boldsymbol{\nabla} \times (\psi \mathbf{e}_\phi)$ in which $\mathbf{e}_\phi$ is a unit vector in the direction of the angle of rotation around the symmetry axis. In other cases a useful representation can be given in the form (see e.g Chandrasekhar 1961, p. 622)

$$\mathbf{A} = \boldsymbol{\nabla} \times (\psi \mathbf{x}) + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\chi \mathbf{x}), \tag{1.1.15}$$

in which ψ and χ are the two *defining scalar* functions. The two terms in this expression are the *toroidal* and *poloidal* components of $\mathbf{A}$, respectively; it is evident that only the second one possesses a radial component. We return to this decomposition in much greater detail in §14.4 and show its usefulness in an example in §7.18.

1.2 The fundamental equations

This book deals with the practice (Part I) and theory (Parts II and III) of solution methods of some fundamental equations which are the most important examples of the three fundamental groups of elliptic, hyperbolic and parabolic equations (see e.g. Courant & Hilbert 1953; Garabedian 1964; Renardy & Rogers 1993 and many others). Here we summarize these equations but, first, we establish some terminology related to boundary and initial conditions.

1.2.1 Boundary conditions

There is a standard terminology to refer to the types of space-time boundary conditions (which would be initial or "final" conditions for the time variable) normally associated with second-order equations:

- *Dirichlet data*: The function is prescribed (e.g. equal to 0) on the boundary;
- *Neumann data*: The normal gradient of the function is prescribed (e.g. equal to 0) on the boundary. With this type of boundary condition, if only derivatives of the unknown function appear in the governing equation, the solution cannot be unique as any function differing by a constant from a solution of the problem is also a solution;
- *Cauchy data*: Both the function and its normal gradient are prescribed on the boundary;

- *Mixed data*: A linear combination of the unknown function and its normal gradient is prescribed on the boundary;
- Conditions may also be of one type (e.g., Dirichlet) on part of the boundary and of another type (e.g. Neumann) on other parts. For example, Neumann conditions might be specified on the finite part of the boundaries, and the Dirichlet-type condition $u \to 0$ at infinity may be added to make the solution unique.

The nature of the boundary conditions to be associated with each specific equation type is quite important and is one aspect of the distinction between properly and improperly posed problems. Although we do not deal with the general theory, we illustrate this point with an example in §4.7.

1.2.2 Elliptic equations

Generally speaking, elliptic equations describe a situation of equilibrium established a long time after the beginning of a process: all forces have equilibrated, all disturbances have either damped out or propagated to infinity and so forth. The most fundamental equation of this type is the *Laplace equation*

$$\nabla^2 u = 0. \tag{1.2.1}$$

Functions satisfying this equation in some spatial region Ω are termed *harmonic* in Ω. The non-homogeneous form of this equation is known as the *Poisson equation*:[4]

$$-\nabla^2 u = f, \tag{1.2.2}$$

in which the function $f(\mathbf{x})$, representing distributed sources of the field u, is given.

Another equation of the same mathematical type which is frequently encountered is the *Helmholtz equation* or *reduced wave equation*

$$\nabla^2 u + k^2 u = -f(\mathbf{x}), \tag{1.2.3}$$

in which both f and the (real or complex, positive or negative) parameter k^2 are given. As will be seen in several examples, when the parameter k^2 is real and positive, there are some very significant qualitative differences between the solutions of (1.2.1) or (1.2.2) and those of (1.2.3).

It will be clear from the physical considerations described in the following sections that the boundary conditions to be associated with these equations cannot be of the Cauchy type. An attempt to prescribe such conditions would result in an ill-behaved and physically unacceptable solution (see example in §4.7). Appropriate boundary conditions are of the Dirichlet, Neumann, or mixed type.

There are important types of higher-order equations of the same mathematical type, chief among them the *biharmonic equation*:

$$\nabla^4 u = \nabla^2 \left(\nabla^2 u \right) = 0, \tag{1.2.4}$$

[4] A physical justification for the minus sign is that an integration by parts of $-u\nabla^2 u$ causes the appearance of the non-negative-definite quantity $|\boldsymbol{\nabla} u|^2$ which often has the physical meaning of an energy density.

which is efficiently split into the Laplace–Poisson system (§7.7)

$$\nabla^2 v = 0, \qquad \nabla^2 u = v. \tag{1.2.5}$$

1.2.3 Hyperbolic equations

Hyperbolic equations generally describe propagation with a finite speed. The prototypical equation of this type is the scalar *wave equation*

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = F(\mathbf{x}, t), \tag{1.2.6}$$

in which the constant c is the speed of propagation of the wave and F, representing distributed sources of the field associated with the wave, is given. By referring, for instance, to a mechanical problem in which u might represent a displacement caused by the wave, it is intuitively clear that, on the "time boundary" $t = 0$, conditions of the Cauchy type, namely

$$u(\mathbf{x}, t = 0) = u_0(\mathbf{x}), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = v(\mathbf{x}), \tag{1.2.7}$$

would be appropriate. On the spatial boundaries through which the wave enters the domain of interest either Dirichlet, Neumann or mixed conditions can be applied.

Particularly simple solutions of the homogeneous wave equation have the form

$$u(\mathbf{x}, t) = \Phi\left(\mathbf{e}\cdot\mathbf{x} - ct\right), \tag{1.2.8}$$

where $\mathbf{e}$ is a unit vector and Φ an arbitrary function admitting a (conventional or distributional) double derivative. It is evident that u as given by this formula has a constant value on the planes $\mathbf{e}\cdot\mathbf{x} - ct = \text{const.}$, which justifies the denomination *plane wave* given to solutions of this type. The normal to this family of planes is parallel to $\mathbf{e}$ and identifies the direction of propagation. In one space dimension only two such waves are possible, one propagating to the left and one to the right so that the most general solution of (1.2.7) has the well-known d'Alembert form

$$u(x, t) = \Phi_+(x - ct) + \Phi_-(x + ct). \tag{1.2.9}$$

In spaces of higher dimensionality a much greater variety of plane waves is possible and any arbitrary superposition of them is also a solution of the equation (see John 1955).

When there is a steady boundary or volume source maintaining monochromatic waves, i.e. waves of a single frequency $\omega/2\pi$ (ω being the angular frequency), and all transients have died out, one expects the solution to have the form

$$u(\mathbf{x}, t) = v(\mathbf{x})\cos\left[\omega t + \psi(\mathbf{x})\right], \tag{1.2.10}$$

i.e., to consist of monochromatic waves of the same frequency with a spatially modulated amplitude v and phase ψ. It is convenient to express this functional form as $u(\mathbf{x}, t) = w(\mathbf{x})e^{-i\omega t}$ allowing w to be complex and tacitly agreeing to take the real part of the complex

expression. When this functional form is substituted into (1.2.6) with F assumed to have the form $F(\mathbf{x}, t) = f(\mathbf{x})e^{-i\omega t}$, w is found to satisfy

$$\nabla^2 w + \frac{\omega^2}{c^2} w = -f, \tag{1.2.11}$$

i.e., the Helmholtz equation (1.2.3) with $k^2 = \omega^2/c^2$; in this context, k is the *wave number*, equal to 2π divided by the wavelength of the monochromatic wave. The same association between the wave and Helmholtz equations is encountered if (1.2.6) is solved by the Fourier transform in the time variable.

If there is no agent to sustain waves and all those initially present have propagated away from the region of interest, the solution of (1.2.6) becomes independent of time and the equation reduces then to the Laplace (1.2.1) or Poisson (1.2.2) forms.

In some cases the effect of damping modifies (1.2.6) to the form

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{2a}{c^2}\frac{\partial u}{\partial t} - \nabla^2 u = F(\mathbf{x}, t), \tag{1.2.12}$$

with $a \geq 0$, which is known as the *telegrapher equation*. An example is given in §1.5.

In electromagnetism and elasticity the propagating quantities are vectors rather than scalars. In these cases it is preferable to write the wave equation using the expression (1.1.10) for the Laplacian to put the equation in the coordinate-invariant form

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{A} = \mathbf{F}, \tag{1.2.13}$$

where it has been assumed that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ as often happens, or can be caused to happen by means of a suitable gauge transformation. In this case, plane wave solutions of the homogeneous equation have the form

$$\mathbf{A} = (\mathbf{b} \times \mathbf{e})\, \Psi\, (\mathbf{e} \cdot \mathbf{x} - ct)\,, \tag{1.2.14}$$

in which $\mathbf{b}$ is an arbitrary constant vector and Ψ a function. When $\mathbf{A}$ is a vector potential, the quantity with a physical meaning is $\boldsymbol{\nabla} \times \mathbf{A} = [\mathbf{b} - (\mathbf{b} \cdot \mathbf{e})\mathbf{e}]\Psi'$, which is perpendicular to $\mathbf{e}$. These waves are therefore polarized in the direction perpendicular to the direction of propagation and, for this reason, are termed *transverse*.

1.2.4 Parabolic equations

The prototypical parabolic equation is the *diffusion equation*

$$\frac{\partial u}{\partial t} = D\nabla^2 u + f(\mathbf{x}, t), \tag{1.2.15}$$

in which the constant D, with dimensions of (length)^2/time, is the *diffusivity* or diffusion coefficient of the quantity u and $f(\mathbf{x}, t)$ is given. The introduction of a scaled time $t_* = Dt$ with $f_* = f/D$ permits us to consider the equation with $D = 1$, which we will often do without explicitly mentioning that a scaled time variable is being used. For these equations initial conditions

$$u(\mathbf{x}, t = 0) = u_0(\mathbf{x}) \tag{1.2.16}$$

are appropriate, together with Dirichlet, Neumann or mixed conditions on the spatial boundaries.

For steady conditions the diffusion equation degenerates to the Laplace (1.2.1) or Poisson (1.2.2) forms. If (1.2.15) is solved by a Fourier or Laplace transform in time, the transformed function satisfies the Helmholtz equation (1.2.3) with k^2 negative or complex (see §5.5).

In diffusion problems it is possible that the source f, rather than being prescribed a priori, is a function of u describing, e.g., the disappearance of u due to a chemical reaction. When this dependence is linear we have

$$\frac{\partial u}{\partial t} = D\nabla^2 u - \alpha u. \tag{1.2.17}$$

If $\alpha = \text{const.}$ the simple substitution $u = e^{-\alpha t} v$ brings the equation into the standard form (1.2.15), written for v, with $f = 0$.

1.3 Diffusion

The most elementary balance relation states that the rate of change of the total amount of some quantity distributed with a density U within a fixed volume V equals the net transport through the boundary S of V plus the contribution of the sources q inside the volume:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V U \,\mathrm{d}V = -\oint_S \mathbf{Q}\cdot\mathbf{n}\,\mathrm{d}S + \int_V q\,\mathrm{d}V\,, \tag{1.3.1}$$

in which $\mathbf{Q}$ is the flux of U and $\mathbf{n}$ the outward unit normal. By applying the divergence theorem (p. 400), in view of the arbitrariness of the control volume V, the previous equation implies that, almost everywhere in space,[5]

$$\frac{\partial U}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{Q} + q. \tag{1.3.2}$$

In heat transfer (1.3.1) derives from the first principle of thermodynamics for an incompressible medium: U is the enthalpy per unit volume, $\mathbf{Q}$ the heat flux and q the internal heat generation rate per unit volume. If the medium has constant properties, $U = \rho c_p T$, with ρ the density, c_p the specific heat and T the temperature. The heat flux $\mathbf{Q}$ must depend on T in such a way that it vanishes when T is spatially uniform. For an isotropic medium the simplest functional form satisfying this condition is embodied in *Fourier's law of conduction*:

$$\mathbf{Q} = -k\boldsymbol{\nabla}T, \tag{1.3.3}$$

in which the thermal conductivity k must be positive for heat to flow away from the hotter regions. A relation of this type may be seen as a truncated Taylor series expansion of a more

[5] The qualification "almost everywhere" has the technical meaning "aside from a set of zero measure"; see p. 684.

general constitutive relation. This is a typical mechanism through which the divergence of a flux gives rise to the Laplacian operator. Upon substitution into (1.3.2) we have

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\nabla^2 T + \frac{q}{\rho c_p}, \tag{1.3.4}$$

which has the form (1.2.15) with $D = k/\rho c_p$. Because of this derivation one often refers to this latter equation as to the heat, or conduction, equation.

In steady conditions and in the absence of sources, (1.3.4) reduces to $\nabla^2 T = 0$ so that the temperature field is harmonic and $\oint \mathbf{n}\cdot\boldsymbol{\nabla} T \, \mathrm{d}S = 0$ from (1.1.3). The underlying physical picture permits us to give a physically compelling justification of the *maximum principle* for harmonic functions, according to which a harmonic function defined in a region Ω cannot attain a local maximum or minimum inside Ω but only on the boundary (see p. 76). Indeed, if T had e.g. a maximum at a certain point interior to Ω, on a sufficiently small surface surrounding this point heat would flow away so that $\mathbf{n}\cdot\boldsymbol{\nabla} T$ would be negative everywhere and the integral could not vanish.

Solution of the diffusion equation clearly requires that an initial temperature distribution $T(\mathbf{x}, t = 0)$ be known. In physical terms, it is evident that the solution will be affected by conditions at the spatial boundaries: the imposition of a prescribed temperature (Dirichlet condition) or a prescribed heat flux (Neumann condition) will certainly affect the spatial and temporal evolution of the temperature field. If the surface of the medium is in contact with a moving fluid capable of removing heat at a rate $h(T - T_\infty)$, where h is a heat transfer coefficient and T_∞ a constant ambient temperature, continuity of heat flux at the surface results in

$$-k\mathbf{n}\cdot\boldsymbol{\nabla} T = h(T - T_\infty), \tag{1.3.5}$$

which is a condition of the mixed type for the unknown $u = T - T_\infty$. A similar condition is approximately valid if the surface of the medium exchanges radiant energy with its surroundings. According to the Stefan–Boltzmann law, a surface emits radiant energy at a rate $\varepsilon\sigma T^4$ per unit area, where ε is the surface emissivity and σ is the Stefan–Boltzmann constant. The incident energy absorbed from the surroundings at temperature T_∞ is $\alpha\sigma T_\infty^4$, with α the surface absorptivity. From Kirchhoff's law, $\alpha = \varepsilon$ and, therefore, continuity of heat flux across the surface requires that

$$-k\mathbf{n}\cdot\boldsymbol{\nabla} T = \varepsilon\sigma(T^4 - T_\infty^4) = \varepsilon\sigma\frac{T^4 - T_\infty^4}{T - T_\infty}(T - T_\infty) \simeq 4\varepsilon\sigma T_\infty^3(T - T_\infty), \tag{1.3.6}$$

provided the temperature difference is not too large. For this reason, the denomination *radiation condition* is encountered in the heat transfer literature to denote conditions of the mixed type.

Another physical process resulting in an equation of the type (1.3.1) is the diffusion of a solute in a solvent. In this case we may take the mass density ρ_d of the diffusing substance as the conserved quantity U in (1.3.1). The flux $\mathbf{Q}$ is given by Fick's law of diffusion as $\mathbf{Q} = -\rho\mathscr{D}\boldsymbol{\nabla}(\rho_d/\rho)$, in which ρ is the total mass density of the solute–solvent mixture and $\mathscr{D}$ the mass diffusivity. When both these quantities are approximately constant $\boldsymbol{\nabla}\cdot\mathbf{Q} \simeq -\rho\mathscr{D}\nabla^2\rho_d$ and the standard form (1.2.15) is recovered. In this case the source term f may represent the generation or disappearance of solute due e.g. to chemical reaction. If