

1 Euclidean geometry

We familiarise ourselves with the axioms of Euclidean geometry in the plane and derive some geometric implications, among them the existence of the parallel line. We briefly discuss the historical importance of the parallel axiom. The existence of the Cartesian model shows the consistency of the Euclidean axioms. The Cartesian model is used to investigate Euclidean trigonometry.

1.1 The axiomatic approach

Geometry is one of the oldest of all sciences. Remarkable geometric knowledge was already present in the advanced oriental cultures of the fifth–third centuries BCE. Practical problems from metrology, architecture, astronomy and navigation were considered on an abstract level and led to geometric laws. For instance, the Egyptians used the formula for the area of a triangle

$$\text{area} = \frac{\text{length of base line} \times \text{height}}{2}$$

and the approximate formula for the area of a circle

$$\text{area} = \left(\text{diameter} - \frac{\text{diameter}}{9} \right)^2.$$

The latter corresponds to an approximation of π by $\frac{256}{81} \approx 3.1605$. No difference was made between exact and approximate formulae in principle. Mathematical knowledge was there in the form of laws, justifications or proofs were not given.

This changed in Greece between 350 and 200 BCE. Aspects of usefulness were then superseded by the desire for understanding. Mathematicians not only wanted to know certain laws, but also why they hold. This was the starting point of the axiomatisation of geometry. At first only a few intuitively evident axioms were laid down, from which it was thought that everything else could be derived logically in a rigorous manner. In what follows we will become

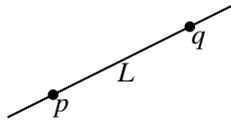
acquainted with the axioms of Euclidean plane geometry and go through some rather simple implications as illustrations of the axiomatic proof. We will mainly follow the formulation of the axioms presented by Hilbert in [13].

The axioms can be classified into five groups. We begin with the incidence axioms. To formulate these, we need two sets \mathcal{P} and \mathcal{G} , whose elements we call points and straight lines respectively. Further assume that for every point $p \in \mathcal{P}$ and every straight line $L \in \mathcal{G}$ the statement “ p is contained in L ”, in symbols “ $p \in L$ ”, is either true or false. Note that the symbol “ \in ” does not denote a set-theoretic inclusion in this case, since the straight lines L are for now not sets, but abstract elements of \mathcal{G} . We nevertheless want to use this suggestive notation. Let us now move on to the first axioms.

Incidence axioms These axioms make some statements about the containedness of points in straight lines.

AXIOM I₁ *For any two points there exists a straight line that goes through both of them,*

$$\forall p, q \in \mathcal{P} \quad \exists L \in \mathcal{G} : \quad p \in L \text{ and } q \in L.$$



AXIOM I₂ *There is at most one straight line through any two distinct points,*

$$\forall p, q \in \mathcal{P}, p \neq q, \quad \forall L, M \in \mathcal{G}, p \in L, q \in L, p \in M, q \in M : \quad L = M.$$

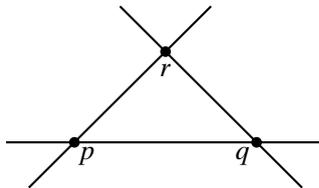
For any two distinct points p and q , there is by those first two axioms exactly one straight line that goes through both of them, we will from now on denote it by $L(p, q)$.

AXIOM I₃ *Every straight line contains at least two distinct points,*

$$\forall L \in \mathcal{G} \quad \exists p, q \in \mathcal{P}, p \neq q : \quad p \in L \text{ and } q \in L.$$

AXIOM I₄ *There exist three points that do not lie on a straight line,*

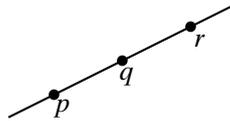
$$\exists p, q, r \in \mathcal{P} : \quad \nexists L \in \mathcal{G} \text{ with } p \in L, q \in L, r \in L.$$



Axiom I₄ expresses that our geometry has at least two dimensions.

Axioms of order For the formulation of these axioms we need, in addition to the notions of \mathcal{P} , \mathcal{G} and \in , that for every triple (p, q, r) of points the statement “ q lies between p and r ” must be either true or false. The following axioms must be satisfied.

AXIOM A₁ *If q lies between p and r , then p , q and r are three pairwise distinct points on a straight line.*



AXIOM A₂ *If q lies between p and r , then q lies between r and p .*

For two points p and q we call the set of all points that lie between p and q the **line segment** from p to q and write \overline{pq} . Axiom A₂ therefore implies $\overline{pr} = \overline{rp}$.

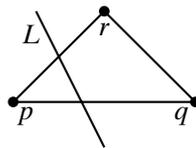
AXIOM A₃ *For any two distinct points p and q there exists a point r , such that q lies between p and r .*

Attention This axiom does not say that for any two given points, there exists another point between them. We will first have to *prove* this, see theorem 1.1.1.

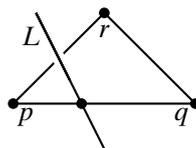
AXIOM A₄ *Given three points, at most one of them lies between the two others.*

If **two straight lines** L and M have a point p in common, $p \in L$ and $p \in M$, then we sometimes say that L and M **intersect**, in symbols $L \cap M \neq \emptyset$. We say that a **line segment** \overline{pr} and a **straight line** L **intersect** if there exists a point q with $q \in L$ between p and r .

AXIOM A₅ *Let p , q and r be three points that do not lie on a straight line, let L be a straight line that does not contain any of these three points. If L intersects the line segment \overline{pq} , then L intersects precisely one of the other two line segments \overline{pr} or \overline{qr} .*



This means that a straight line which enters a triangle must leave it through one of the other two sides. It also illustrates that our geometry does not have more than two dimensions. In three dimensions axiom A₅ would not be valid:



Let us now prove a first theorem, using the axioms stated so far.

Theorem 1.1.1 *For any two distinct points p and q there exists a point r which lies between p and q , i.e. the line segment \overline{pq} is not empty.*

Proof Let p and q be two points. By axiom I_4 there exists a point s that does not lie on the straight line $L(p, q)$. By axiom A_3 there is a point t such that s lies between p and t . Another application of axiom A_3 gives a point u such that q lies between t and u . The straight line $L := L(s, u)$ intersects the line segment \overline{pt} at s .

The point t does not lie on the straight line $L(p, q)$, since otherwise s would by axiom A_1 also lie on that straight line, contradicting the choice of s . We can therefore apply axiom A_5 to the straight line L and the three points p, q and t . As L intersects the line segment \overline{pt} , L must by axiom A_5 also intersect one of the two line segments \overline{pq} or \overline{tq} , unless it contains one of the three points p, q or t .

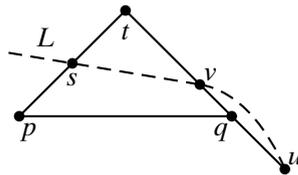
First case L contains p or t .

Then L agrees with the straight line $L(p, t)$ by axiom I_2 . Hence u lies on $L(p, t)$ and axiom A_1 implies that q lies on $L(p, t)$ as well. Hence p, q and t do lie on a straight line, a contradiction.

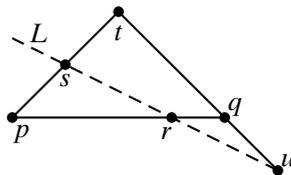
Second case L contains q .

Several applications of axiom I_2 show that the points s, u, q, t and p lie on a straight line, a contradiction.

Third case L intersects the line segment \overline{tq} at a point v .



Then L and $L(t, q)$ have the two points u and v in common. If $u = v$, then u would lie between t and q while q lies between t and u , a contradiction to axiom A_4 . Thus L and $L(t, q)$ have two distinct points in common, and axiom I_2 implies that $L = L(t, q)$. But then both s and p lie on L , i.e. p, q and t lie on a straight line, a contradiction.



Thus L and \overline{pq} must intersect at a point r . In particular, \overline{pq} cannot be empty. □

Exercise 1.1 Let p, q, r and s be points on a straight line. Show that if q lies between p and s and if additionally r lies between q and s , then r lies between p and s as well.

Using this exercise and theorem 1.1.1, it is now easy to solve the following exercise.

Exercise 1.2 Show the following: there is an infinite number of distinct points between two points.

Definition 1.1.2 Let L be a straight line, $p \in L$. Let q and r be two points on L , neither of which is p . We say that q and r **lie on the same side of the point p** if p does not lie between q and r .

Exercise 1.3 Let L be a straight line and let $p \in L$ be a point on L . Show that the relation “ q_1 lies on the same side of p as q_2 ” defines an equivalence relation on the set $\{q \in L \mid q \neq p\}$.

An equivalence class of points on L that do not equal p can then be referred to as a **side of p on L** .

Exercise 1.4 Show that there are exactly two sides of p on L .

Definition 1.1.3 Let L be a straight line and let p and q be two points that do not lie on L . We say that p and q **lie on the same side of a straight line L** if the line segment \overline{pq} does not intersect the straight line L .

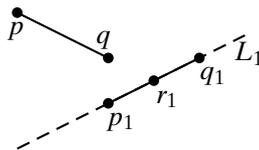
Exercise 1.5 Let L be a straight line. Show that the relation “ q_1 lies on the same side of L as q_2 ” defines an equivalence relation on the set $\{q \mid q \notin L\}$.

Again we can call an equivalence class of points not on L a **side of L** .

Exercise 1.6 Show that there are exactly two sides of L .

Congruence axioms To formulate the third group of axioms, the congruence axioms, we need in addition to the previous notions that for every pair $(\overline{pq}, \overline{p_1q_1})$ of line segments the formal statement “ \overline{pq} is congruent to $\overline{p_1q_1}$ ” is either true or false.

AXIOM K_1 (reproduction of lengths) Let \overline{pq} be a line segment, let L_1 be a straight line, let $p_1, r_1 \in L_1, r_1 \neq p_1$. Then there is a point $q_1 \in L_1$ on the same side of p_1 as r_1 such that \overline{pq} is congruent to $\overline{p_1q_1}$.



In this axiom only the existence of a congruent line segment is required. Its uniqueness needs to be proved later with the aid of other axioms.

AXIOM K_2 *If the line segments $\overline{p_1q_1}$ and $\overline{p_2q_2}$ are both congruent to the line segment \overline{pq} , then $\overline{p_1q_1}$ is congruent to $\overline{p_2q_2}$ as well.*

Four more congruence axioms will follow. We can nevertheless already prove a first implication.

Lemma 1.1.4 *The congruence of line segments defines an equivalence relation on the set of line segments.*

Proof (a) Let \overline{pq} be a line segment. We show that \overline{pq} is congruent to itself. Let L be a straight line that contains p , $p \in L$. By axiom K_1 there exists a point r on L such that \overline{pq} and \overline{pr} are congruent. Then $\overline{p_1q_1} := \overline{p_2q_2} := \overline{pq}$ is congruent to \overline{pr} and hence \overline{pq} is by axiom K_2 congruent to itself.

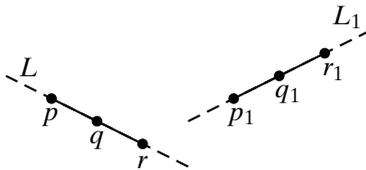
(b) (symmetry) Let \overline{pq} be congruent to $\overline{p_1q_1}$. We show that then $\overline{p_1q_1}$ is congruent to \overline{pq} . It follows from (a) that $\overline{p_1q_1}$ is congruent to $\overline{p_1q_1}$. Axiom K_2 now gives that $\overline{p_1q_1}$ is congruent to \overline{pq} .

(c) (transitivity) If $\overline{p_1q_1}$ is congruent to $\overline{p_2q_2}$ and $\overline{p_2q_2}$ is congruent to $\overline{p_3q_3}$, then we need to show that $\overline{p_1q_1}$ is congruent to $\overline{p_3q_3}$. This follows directly from axiom K_2 together with (b). □

We will from now on sometimes denote “ $\overline{p_1q_1}$ is congruent to $\overline{p_2q_2}$ ” by “ $\overline{p_1q_1} \equiv \overline{p_2q_2}$ ”.

AXIOM K_3 (additivity of line segments) *Let L and L_1 be straight lines, let $p, q, r \in L$ be three pairwise distinct points on L and $p_1, q_1, r_1 \in L_1$ likewise on L_1 . Assume that the line segments \overline{pq} and $\overline{q_1r_1}$ do not have any common points, $\overline{pq} \cap \overline{q_1r_1} = \emptyset$. Analogously let $\overline{p_1q_1} \cap \overline{q_1r_1} = \emptyset$.*

If now $\overline{pq} \equiv \overline{p_1q_1}$ and $\overline{q_1r_1} \equiv \overline{q_1r_1}$, then $\overline{pr} \equiv \overline{p_1r_1}$.



We need the concept of the *angle* for the formulation of the three other congruence axioms.

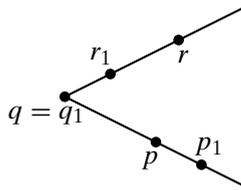
Definition 1.1.5 *An **angle** is an equivalence class of triples of points p, q and r that do not lie on a straight line, where two triples (p, q, r) and (p_1, q_1, r_1) are equivalent if*

- (i) $q = q_1$,
- (ii) $L(p, q) = L(p_1, q)$ and p and p_1 lie on the same side of q ,
- (iii) $L(r, q) = L(r_1, q)$ and r and r_1 lie on the same side of q ,

or if

- (i) $q = q_1$,
- (ii) $L(p, q) = L(r_1, q)$ and p and r_1 lie on the same side of q ,
- (iii) $L(r, q) = L(p_1, q)$ and r and p_1 lie on the same side of q .

For the equivalence class of (p, q, r) we write $\angle(p, q, r)$. The point q is then called the **vertex** of the angle $\angle(p, q, r)$.

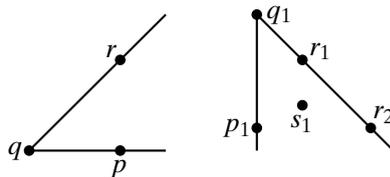


We now additionally require that for any two angles $\angle(p, q, r)$ and $\angle(p_1, q_1, r_1)$ the formal statement “ $\angle(p, q, r)$ is congruent to $\angle(p_1, q_1, r_1)$ ” is either true or false. Again, we write “ $\angle(p, q, r) \equiv \angle(p_1, q_1, r_1)$ ” if $\angle(p, q, r)$ is congruent to $\angle(p_1, q_1, r_1)$.

AXIOM K₄ *The congruence of angles induces an equivalence relation on the set of angles.*

AXIOM K₅ (reproduction of angles) *Let p, q, r be points that do not lie on a straight line, and let p_1, q_1, s_1 be another set of points that do not lie on a straight line. Then there exists a point r_1 on the same side of $L(p_1, q_1)$ as s_1 such that the angle $\angle(p_1, q_1, r_1)$ is congruent to the angle $\angle(p, q, r)$.*

If r_2 is another point with the same properties as r_1 , i.e. r_2 also lies on the same side of $L(p_1, q_1)$ as s_1 and if $\angle(p_1, q_1, r_2) \equiv \angle(p, q, r)$, then $\angle(p_1, q_1, r_1) = \angle(p_1, q_1, r_2)$.

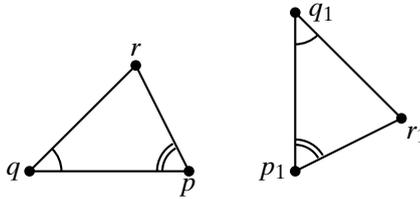


Axiom K₅ says that we can reproduce a given angle in a unique way if we are given the vertex, one line adjacent to the angle and the side of the other line next to the angle.

The last congruence axiom relates the congruence of line segments to that of angles. Until now the two notions of congruence existed entirely separately.

AXIOM K_6 *Let (p, q, r) be a triple of points that do not lie on a straight line and (p_1, q_1, r_1) likewise. If $\overline{pq} \equiv \overline{p_1q_1}$, $\overline{pr} \equiv \overline{p_1r_1}$ and $\angle(q, p, r) \equiv \angle(q_1, p_1, r_1)$, then*

$$\angle(p, q, r) \equiv \angle(p_1, q_1, r_1).$$

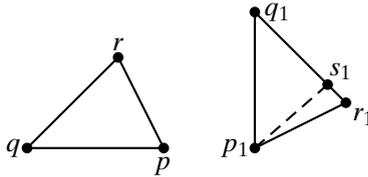


Let us make some inferences using the axioms introduced so far. We will first extend the statement of the last axiom.

Theorem 1.1.6 *Let (p, q, r) be a triple of points that do not lie on a straight line, (p_1, q_1, r_1) likewise. If $\overline{pq} \equiv \overline{p_1q_1}$, $\overline{pr} \equiv \overline{p_1r_1}$ and $\angle(q, p, r) \equiv \angle(q_1, p_1, r_1)$, then*

$$\angle(p, q, r) \equiv \angle(p_1, q_1, r_1), \quad \angle(p, r, q) \equiv \angle(p_1, r_1, q_1), \quad \overline{qr} \equiv \overline{q_1r_1}.$$

Proof The angle congruences follow directly from axiom K_6 , in the second case after renaming the variables. It remains to show that $\overline{qr} \equiv \overline{q_1r_1}$. By axiom K_1 we can find a point s_1 on the straight line $L(q_1, r_1)$ which is on the same side as r_1 and satisfies $\overline{q_1s_1} \equiv \overline{qr}$.



We apply axiom K_6 to (p, q, r) and (p_1, q_1, s_1) and conclude that

$$\angle(q, p, r) \equiv \angle(q_1, p_1, s_1).$$

On the other hand we have $\angle(q, p, r) \equiv \angle(q_1, p_1, r_1)$, and by the uniqueness from axiom¹ K_5

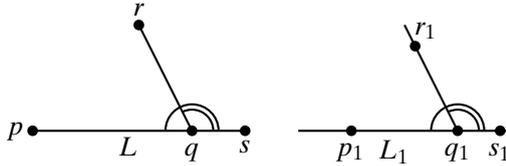
¹ For an application of the uniqueness statement from axiom K_5 it must be ensured that r_1 and s_1 lie on the same side of $L(p_1, q_1)$. This is left as an exercise for the reader.

$$\angle(q_1, p_1, s_1) = \angle(q_1, p_1, r_1).$$

If we now had $r_1 \neq s_1$, then we would conclude that p_1 and q_1 both lie on the straight line $L(r_1, s_1)$, i.e. p_1, q_1 and r_1 would lie on a straight line, which contradicts the assumption.

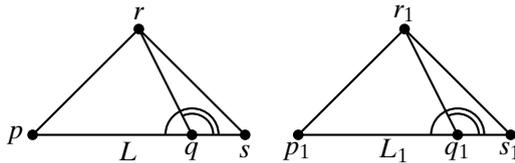
Thus $r_1 = s_1$ and hence $\overline{q_1 r_1} \equiv \overline{q_1 s_1}$. □

Theorem 1.1.7 (Congruence of adjacent angles) *Suppose that the pairwise distinct points p, q and s lie on a straight line L , while $r \notin L$. Analogously, let $p_1, q_1, s_1 \in L_1$ be pairwise distinct, $r_1 \notin L_1$. If $\angle(p, q, r)$ and $\angle(p_1, q_1, r_1)$ are congruent, then the same is true for $\angle(s, q, r)$ and $\angle(s_1, q_1, r_1)$.*



The angle $\angle(s, q, r)$ is sometimes called the **adjacent angle** of $\angle(p, q, r)$. The theorem thus says that adjacent angles of congruent angles are congruent as well.

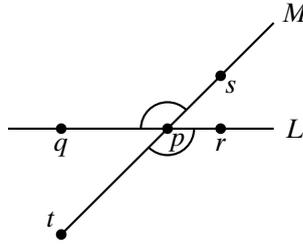
Proof The points p_1, r_1 and s_1 can by axiom K_1 be assumed to have been chosen in such a way that $\overline{p_1 q_1} \equiv \overline{p_1 q}$, $\overline{r_1 q_1} \equiv \overline{r_1 q}$ and $\overline{s_1 q_1} \equiv \overline{s_1 q}$.



From theorem 1.1.6, applied to (p, q, r) and (p_1, q_1, r_1) , it follows that $\overline{p_1 r} \equiv \overline{p_1 r_1}$. By axiom K_3 we then have $\overline{p_1 s} \equiv \overline{p_1 s_1}$. Applying theorem 1.1.6 once again, this time to (r, p, s) and (r_1, p_1, s_1) , we obtain $\overline{r_1 s} \equiv \overline{r_1 s_1}$ and $\angle(q, s, r) \equiv \angle(q_1, s_1, r_1)$. Axiom K_6 then says for (q, s, r) and (q_1, s_1, r_1) that $\angle(s, q, r) \equiv \angle(s_1, q_1, r_1)$. □

Theorem 1.1.8 (Congruence of vertical angles) *Let L and M be two distinct straight lines that intersect at p . Let $r, q \in L$ lie on two distinct sides of p and let $s, t \in M$ lie on two distinct sides of p as well. Then*

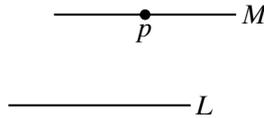
$$\angle(q, p, s) \equiv \angle(r, p, t).$$



Proof Both $\angle(q,p,s)$ and $\angle(r,p,t)$, are adjacent angles of $\angle(q,p,t)$. The angle $\angle(q,p,t)$ is congruent to itself by axiom K_4 . The claim therefore follows by theorem 1.1.7. \square

After these preparations we now come to the first truly interesting geometric theorem.

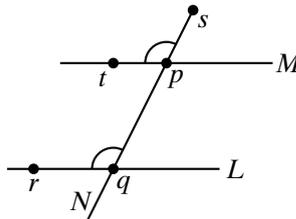
Theorem 1.1.9 (Existence of a parallel) *Let L be a straight line, p a point, $p \notin L$. There then exists a straight line M that contains p and that does not intersect L .*



We then say that M is a **parallel** to L through p . The theorem says that such parallels always exist.

Proof Let L be a straight line and p a point that does not lie on L . We will first construct the line M and then show that it has the desired properties.

For the construction we choose a point $q \in L$ and add the straight line $N := L(p,q)$. We choose another point $r \in L$, $r \neq q$. Then $r \notin N$, since otherwise $p \in L(p,q) = L(q,r) = L$. We reproduce angle $\angle(r,q,p)$ as in axiom K_5 on the straight line N at the point p , i.e. we find points $s \in N$ and $t \notin N$ on the same side of N as r , such that the angle $\angle(t,p,s)$ is congruent to the angle $\angle(r,q,p)$. We now set $M := L(p,t)$.



It remains to show that L and M do not intersect. Suppose that L and M did intersect at a point u . We restrict ourselves to the case that u lies on the same side of N as r and t . The other case is dealt with in a similar way.