A Theory of Fairness and Social Welfare

The definition and measurement of social welfare have been a vexed issue for the past century. This book makes a constructive, easily applicable proposal and suggests how to evaluate the economic situation of a society in a way that gives priority to the worse-off and that respects each individual’s preferences over his or her own consumption, work, leisure, and so on. This approach resonates with the current concern to go “beyond the GDP” in the measurement of social progress. Compared to technical studies in welfare economics, this book emphasizes constructive results rather than paradoxes and impossibilities, and shows how one can start from basic principles of efficiency and fairness and end up with concrete evaluations of policies. Compared to more philosophical treatments of social justice, this book is more precise about the definition of social welfare and reaches conclusions about concrete policies and institutions only after a rigorous derivation from clearly stated principles.

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To Christine and Hélène,

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Preface

The idea of the line of research presented in this book emerged from a conversation in the French foothills of the Pyrenees in April 1996. The prevailing separation between the theory of social choice, the theory of fair allocation, and public economics seemed to call for a unified approach that would construct social preferences similar to the first, relying on fairness concepts from the second, and derive policy conclusions for the third. It took more time than we expected to transform this idea into a recognized theory, and we are pleased to acknowledge the support and constructive influence of many colleagues and friends. The late Louis Gevers, who had worked himself at the intersection of social choice and fair allocation, was the first to express interest and provide encouragements. Our debt to him is immense. Peter Hammond, Philippe Mongin, Juan de Dios Moreno-Ternero, Erwin Ooghe, Erik Schokkaert, Yves Sprumont, Kotaro Suzumura, and Koichi Tadenuma were very helpful at different stages of our work and became coauthors and partners in this research. We are also deeply indebted to Eric Maskin, John Roemer, and John Weymark, who commented on and discussed several of our early works on this topic. Their support was both helpful and encouraging. The interest shown by students who made their own contribution to this direction of research, Efthymios Athanasiou and Giacomo Valletta, was a great reward. The confirmation that a variety of empirical applications were possible came thanks to the fruitful collaboration with Koen Decancq, Brigitte Dormont, Michel Fouquin, Guillaume Gaulier, Stéphane Luchini, Juan de Dios Moreno-Ternero, Christophe Muller, Esther Regnier, and Erik Schokkaert, and the support of CEPII, the French Ministry of Social Affairs, the Junta de Andalucía, and the Foundation of Risk. We also benefited a great deal from the influence, through their work and through stimulating conversation, of Serge Kolm, Hervé Moulin, and William Thomson, as well as Dilip Abreu, Claude d’Aspremont, Tony Atkinson, Salvador Barberà, Charles Blackorby, Walter Bossert, Antoine d’Autume, Angus Deaton, David Donaldson, Rodolphe Dos Santos Ferreira, Lionel Fontagné, Nicolas Gravel, Faruk Gul, Michel Le Breton, Kevin Roberts, Amartya Sen, Alain Trannoy, and Bertil Tungodden.
Preface

This manuscript benefited from many sorts of help. The first chapters owe much to the hospitality of Nuffield College, Oxford, and later chapters to a Lachmann fellowship at the London School of Economics and Political Science, London, and a sabbatical semester at Northwestern University, Evanston. The final preparation of the manuscript was greatly enhanced by a CORE Prize and the focused settings of Louvain-la-Neuve. Summer schools at the Urrutja Foundation, San Sebastian, the University of Hitotsubashi, Tokyo, and the Universities of Málaga, Siena, Strasbourg, and Rouen, provided opportunities for presentation and discussions. Detailed written comments were generously provided by Juan de Dios Moreno-Ternero and Giacomo Valletta, as well as by a reading group comprising Paolo Brunori, Karen Decancq, Koen Decancq, Xavier Jara, Marco Mantovani, Erwin Ooghe, Paolo Piacquadio, Christelle Sapata, and Stéphane Zuber. Two referees made very helpful suggestions and provided detailed remarks that improved the text substantially. The interest expressed, and advice provided, by the two editors who handled the manuscript, Matt Jackson and George Mailath, is also gratefully acknowledged. Scott Parris, from Cambridge University Press, gave us much help with his characteristic enthusiasm. Our colleagues at CERSES and CORE deserve many thanks for providing a supportive environment. Last, but not least, our families have not only accepted absences and absent-mindedness, but also steadily supported our interest in fairness and social justice. We dedicate this book to them.
Introduction

The evaluation of allocations of resources, distributions of well-being, and public policies is a pervasive need in economics and a frequent activity of economists. This is, however, generally considered a difficult and hazardous exercise. The danger is not just the risk of mixing value judgments and factual assessments. It is also, perhaps primarily, the risk of getting lost in the morass of the controversies and impossibility theorems of the field that one can broadly call normative economics. However, the development of this field in the past century has been impressive, and has provided very powerful analytical tools. In particular, the theory of social choice and the theory of fair allocation have, separately, proposed an array of promising concepts and methods. In this book we put the concepts of these two theories to use and propose a general theory of social criteria for economic allocation problems.

In a nutshell, then, this book studies the elaboration of criteria for the evaluation of social and economic situations, and the application of such criteria to the search for optimal public policies. Several broad objectives are assigned to the criteria developed in our analysis.

First, the criteria should be sufficiently comprehensive so when they declare one situation to be preferable to another, there is a sense in which this evaluation takes account of all relevant considerations and is not merely a judgment that the considered situation is better in only one respect. More specifically, the idea is that the criteria should incorporate principles of efficiency as well as principles of equity. Restricting attention to efficiency only, or to equity only, would not provide very useful criteria in our opinion. A related tenet of this requirement of comprehensiveness is that the criteria must be individualistic in the sense of taking account of every individual situation in its own right, and of giving due consideration to every individual’s perspective on his or her own position. Criteria that rely directly on global quantities of the population without grounding this on an assessment of every individual situation are therefore excluded from the outset, although, obviously, it will be considered

1 See, e.g., Arrow (1963) for a criticism of pure efficiency criteria.
quite valuable when, from a properly individualistic criterion, we are sometimes able to derive simple criteria based on global data.

Second, the criteria should be fine-grained enough to be useful in most contexts of public decision making and institutional design. Criteria that simply point to the optimal solution in some special context of implementation are not sufficient from this standpoint. We are looking for sufficiently precise rankings of all the options that may be on the agenda in various economic and political contexts. This may be slightly more than what is really needed in practice, because decision makers need to know only what the optimal option is in their particular context, not how to rank all the suboptimal options or the infeasible options. Given the great variety of possible contexts, however, and, in particular, having in mind that the job of public decision makers is mostly to evaluate imperfect reforms in characteristically suboptimal situations, we think that the most convenient kind of criterion is a fine-grained ranking of all options that may be on the agenda in some possible context.

Finally, because we think of applications in public economics, the criteria must be relevantly applicable to the evaluation of social and economic situations – that is, primarily, the evaluation of conflicts of interest between individuals in the allocation of economic resources. We believe that for such applications, general abstract criteria will not sufficiently take account of the ethically relevant features of individuals’ interests. More precisely, requirements of fairness are sometimes general, but are also often particular to the situation at hand. Fairness in the distribution of unproduced commodities is not the same as fairness in production, and fairness in the production of a private good is not the same as that for a public good. As a consequence, the main focus here is on economic models of resource allocation, rather than on abstract models of political or collective decision, although some words will be said on the latter. What qualifies as a fair solution to one specific allocation problem, however, may depend on how related problems are solved. For instance, the solutions to the public good problem discussed in Chapter 8 make sense only if the income distribution itself is fair.

One defining feature of our analysis throughout this book is that individual situations are not described initially by interpersonally comparable measures of utility or well-being. It is part of our theory to construct interpersonal comparisons on the basis of ordinal noncomparable preferences over bundles of resources. Moreover, we put special emphasis on cases in which preferences are heterogeneous. Some words of explanation are needed about this particular feature of our approach.

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2 In a more philosophical context, Sen (2009) emphasizes that, in view of the imperfections of the societies with which analysts and decision makers are grappling, it is important to rank the social alternatives instead of merely delineating a perfectly just but unattainable society as is done in many theories of justice.
Following Robbins and Samuelson, an important stream of economic analysis has embraced the view that utility cannot easily be compared across individuals, and that interpersonal utility comparisons always involve value judgments with little or no objective basis. In this vein, many economists adopted the ethical assumption that interpersonal comparisons should be made in terms of mental states (such as happiness) and the empirical assumption that mental states are not observable. They therefore deleted utility numbers from their analyses as much as technically possible, focusing on ordinal preferences. The ordinal approaches to welfare economics that are based on the Pareto efficiency criterion and its extensions through compensation tests, including the theory of fair allocation, that somehow emerged from the theory of general equilibrium, have been motivated by such skepticism about interpersonal comparisons of utility.

Although our approach is compatible with this traditional defiance about utilities, we do not endorse the views just described. The behaviorist presumption that mental states, unlike choices of objects and consumption bundles, are not observable has now fallen into disrepute. Mental states such as happiness are amenable to some kind of objective measurement, or are likely to become so in the near future. It is true, nonetheless, that individuals’ various goals in life, and their corresponding achievements, are essentially incommensurable, so measuring satisfaction (a complex judgment that is quite distinct from simple mental states such as happiness or even the feeling of being satisfied) in a meaningful and interpersonally comparable way is problematic. If interpersonal comparisons had to be made in terms of satisfaction, there would indeed be a serious difficulty whenever individual preferences are heterogeneous, as in our models.

More important, however, the ethical assumption that interpersonal comparisons should be made in terms of mental states, which owes much to the utilitarian tradition, is now under heavy criticism. Many authors, especially Rawls (1971, 1982), Dworkin (2000), and Sen (1992), have argued that for the evaluation of social and economic allocations, a focus on subjective or mental states is not appropriate. They argue that social justice deals primarily with the distribution of resources and means of flourishing (including personal characteristics that may be registered as internal resources) rather than the distribution of subjective satisfaction. The way in which individuals obtain degrees of satisfaction from given amounts of resources or capabilities should, in this view, be considered a matter of personal responsibility. We adopt this view in this book, and the metric of interpersonal comparisons that will be used here will be primarily a resource metric. However, unlike Rawls and Sen, we do

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3 See Robbins (1932), Samuelson (1947), and, for a historical synthesis on the origins of positivism in economics, Hammond (1991).

4 To be precise, we should say that comparisons are made in our work in terms of the objects of individual preferences. In our economic models, such objects are resources. But if the objects
not accept the idea that a simple index that disregards individual subjectivity can be used, because we consider it important to take account of individual preferences to allocate resources appropriately. Comparisons must be made in terms of a personalized measure of the value of resources, and devising such a measure is an essential task in our work.

In summary, our approach in terms of ordinal noncomparable preferences over bundles of resources is compatible with several important ethical views, not just with the positivist tradition of ordinalism that has been influential in economics. Obviously, being able to devise social criteria on the sole basis of data that can be extracted from observable demand behavior, or even less, as we shall see, is indeed a practical advantage. But this advantage does not determine our methodology.

The topic of this book can, then, be more technically described as the aggregation of conflicting individual preferences into a consistent social ranking. Admittedly, we are a far cry from achieving the ultimate goal of devising a set of criteria for the evaluation of general social and economic situations, where by “general” we mean a complete description of societies in all dimensions and all details. What we provide here is a series of analyses for simple contexts depicted by tractable models.

The first chapter of this book maps out the field of welfare economics and social choice theory and explains how our approach relates to, and supplements, the existing approaches. Social choice theory has been dubbed the “science of the impossible,” and we explain in particular why, in our view, there is much room for interesting possibilities. It is now well known that obtaining possibility results has to do with the information that is used by the criteria. The fourth chapter examines the informational basis of our approach with greater detail, after some general results of the approach have been presented in the second and third chapters.

Our focus in the first part of the book is the canonical model of distribution of unproduced goods. This model is simple but useful as a basic tool for the analysis of multidimensional problems. Some of the results we obtain with it are recurrent in all contexts. Moreover, it is sufficiently abstract to be versatile, and some results can be easily transposed to other contexts – for instance, when goods are replaced by functionings in the description of individual situations.

The second and third chapters present two basic results that are pervasive in our approach, as they come up in some way or other in all economic models that have been studied so far. The first basic result is a conflict between the idea of reducing resource inequalities across individuals and the Pareto principle. This efficiency–equality tension is due to the fact that with heterogeneous preferences, resource inequalities do not always obviously translate into inequalities in the relevant interpersonally comparable measure.
Introduction

The second basic result is that the combination of the Pareto principle and some mild requirements that impose a minimal inequality aversion (namely, it must be positive, or even simply nonnegative) force the social criteria to actually have an infinite aversion to inequality, as in the maximin criterion. The literature on social welfare contains justifications of the maximin and the leximin criteria that involve rather strong egalitarian requirements, in the one-dimensional context when individual well-being is measured by an interpersonally comparable index of income or utility. The different justification we obtain here hinges on the multidimensional context of multiple goods being allocated among individuals.

The second part of the book examines the particular social rankings that can be defined for the model of distribution of unproduced goods. It considers, in turn, the case of divisible goods and the case of indivisibles. The third part introduces production, for the relatively simple case in which one output is produced with one input, such as labor. We do, however, examine in detail the case of unequal skills, which is particularly relevant for applications to public economics. As alluded to previously, the main value of defining fine-grained rankings of all allocations is the possibility of giving policy advice under any restriction of the set of feasible allocations. A particularly relevant context of application is provided by incentive constraints that arise when the public authority has imperfect information about individual characteristics. We show, in particular, how the social rankings obtained can be used for the evaluation of income tax schedules, when the population is heterogeneous in both skill levels and preferences about leisure and consumption, and such characteristics are private knowledge. This study of production deals with what is technically described as the production of a private good, but we also examine the problem of production of a public good, which is also relevant to public economics. An example of application to public good funding in the second-best context (i.e., when individuals may misrepresent their willingness to pay for the public good) is provided.

In the second and third parts of the book we adopt the same methodology, which consists of defining efficiency and equity requirements and determining what kind of social rankings satisfy these requirements. Once a social ranking is obtained, it can be used for the evaluation of public policies; in the last part of the book, we focus particularly on the translation of the ranking of allocations into a ranking of policies, for standard tax-and-transfer instruments.

We end this introduction with a caveat. This is a work in normative economics, in which we derive social criteria from basic ethical principles and apply them to policy issues. We consider that the role of the economist in this kind of analysis is to establish the link between value judgments and policy conclusions, not to use the authority of expertise to promote personal prejudice. As an illustration of this stance, we often end up considering different criteria that rely on alternative ethical principles. We do not endorse each and every criterion that is proposed here, which would be inconsistent, and we refrain, as much as we can, from expressing definite preferences when several criteria
are on the table. Of course, we exercised some judgment in the selection of the basic principles, retaining (or focusing on) those that appear reasonable for current prevailing views. All in all, we find support in Samuelson’s defense of welfare economics:

It is a legitimate exercise of economic analysis to examine the consequences of various value judgments, whether or not they are shared by the theorist (1947, p. 220).

**How to read this book.** The bulk of the argument in this book requires only some minimal mathematical competence, and our hope is that it is accessible to most economists. However, economic allocations are complex objects and our proofs often involve the examination of several different allocations that differ from each other in all sorts of ways. As a consequence, many of the long and tedious proofs of our results have been relegated to the appendix, in which case the main text contains only an intuitive explanation of the logic of the argument. Among other things, the index lists all the axioms that are used in the search of social criteria, to make it easy to locate their first appearance in the book. The same has been done for the mathematical notations.

**General notations.** The set of real numbers (respectively, non-negative, positive real numbers) is $\mathbb{R}$ (respectively, $\mathbb{R}_+$, $\mathbb{R}_{++}$) the set of natural integers (respectively, relative, positive integers) is $\mathbb{N}$ (respectively, $\mathbb{Z}$, $\mathbb{Z}_{++}$), the set of rational numbers (respectively, positive rational numbers) is $\mathbb{Q}$ (respectively, $\mathbb{Q}_{++}$).

Vector inequalities are denoted $\geq$, $>$, $\gg$. Weak (respectively, strict) set inclusion is denoted $\subseteq$ (respectively, $\subset$).

The cardinality of set $A$ is denoted $|A|$. The set $A^B$ is the set of mappings from $B$ to $A$. The (Minkowski) addition of sets is defined as $A + B = \{x \mid \exists (a, b) \in A \times B, x = a + b\}$.

An ordering is a reflexive and transitive binary relation on a set. The subset of maximal elements of a set $A$ for an ordering $R$ is denoted $\max_R A$. 