Curved Spaces

This self-contained textbook presents an exposition of the well-known classical two-dimensional geometries, such as Euclidean, spherical, hyperbolic and the locally Euclidean torus, and introduces the basic concepts of Euler numbers for topological triangulations and Riemannian metrics. The careful discussion of these classical examples provides students with an introduction to the more general theory of curved spaces developed later in the book, as represented by embedded surfaces in Euclidean 3-space, and their generalization to abstract surfaces equipped with Riemannian metrics. Themes running throughout include those of geodesic curves, polygonal approximations to triangulations, Gaussian curvature, and the link to topology provided by the Gauss–Bonnet theorem.

Numerous diagrams help bring the key points to life and helpful examples and exercises are included to aid understanding. Throughout the emphasis is placed on explicit proofs, making this text ideal for any student with a basic background in analysis and algebra.

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Curved Spaces

From Classical Geometries to Elementary Differential Geometry

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For Stanzi, Toby and Alexia,
in the hope that one day
they might understand what is written herein,
and to Sibylle.

Contents

Preface

This book represents an expansion of the author's lecture notes for a course in Geometry, given in the second year of the Cambridge Mathematical Tripos. Geometry tends to be a neglected part of many undergraduate mathematics courses, despite the recent history of both mathematics and theoretical physics being marked by the continuing importance of geometrical ideas. When an undergraduate geometry course is given, it is often in a form which covers various assorted topics, without necessarily having an underlying theme or philosophy — the author has in the past given such courses himself. One of the aims in this volume has been to set the well-known classical two-dimensional geometries, Euclidean, spherical and hyperbolic, in a more general context, so that certain geometrical themes run throughout the book. The geometries come equipped with well-behaved distance functions, which in turn give rise to curvature of the space. The curved spaces in the title of this book will nearly always be two-dimensional, but this still enables us to study such basic geometrical ideas as geodesics, curvature and topology, and to understand how these ideas are interlinked. The classical examples will act both as an introduction to, and examples of, the more general theory of curved spaces studied later in the book, as represented by embedded surfaces in Euclidean 3-space, and more generally by abstract surfaces with Riemannian metrics.

The author has tried to make this text as self-contained as possible, although the reader will find it very helpful to have been exposed to first courses in Analysis, Algebra, and Complex Variables beforehand. The course is intended to act as a link between these basic undergraduate courses, and more theoretical geometrical theories, as represented say by courses on Riemann Surfaces, Differential Manifolds, Algebraic Topology or Riemannian Geometry. As such, the book is not intended to be another text on Differential Geometry, of which there are many good ones in the literature, but has rather different aims. For books on differential geometry, the author can recommend three in particular, which he has consulted when writing this volume, namely [5], [8] and [9]. The author has also not attempted to put the geometry he describes into a historical perspective, as for instance is done in [8].

As well as making the text as self-contained as possible, the author has tried to make it as elementary and as explicit as possible, where the use of the word elementary here implies that we wish to rely as little as possible on theory developed elsewhere. This explicit approach does result in one proof where the general argument is both intuitive and clear, but where the specific details need care to get correct, the resulting formal proof therefore being a little long. This proof has been placed in an appendix to Chapter 3, and the reader wishing to maintain his or her momentum should skip over this on first reading. It may however be of interest to work through this proof at some stage, as it is by understanding where the problems lie that the more theoretical approach will subsequently be better appreciated. The format of the book has however allowed the author to be more expansive than was possible in the lectured course on certain other topics, including the important concepts of differentials and abstract surfaces. It is hoped that the latter parts of the book will also serve as a useful resource for more advanced courses in differential geometry, where our concrete approach will complement the usual rather more abstract treatments.

The author wishes to thank Nigel Hitchin for showing him the lecture notes of a course on Geometry of Surfaces he gave in Oxford (and previously given by Graeme Segal), which will doubtless have influenced the presentation that has been given here. He is grateful to Gabriel Paternain, Imre Leader and Dan Jane for their detailed and helpful comments concerning the exposition of the material, and to Sebastian Pancratz for his help with the diagrams and typesetting. Most importantly, he wishes to thank warmly his colleague Gabriel Paternain for the benefit of many conversations around the subject, which have had a significant impact on the final shape of the book.