ADVANCED MAGNETOHYDRODYNAMICS With Applications to Laboratory and Astrophysical Plasmas

Following on from the companion volume *Principles of Magnetohydrodynamics*, this textbook analyzes the applications of plasma physics to thermonuclear fusion and plasma astrophysics from the single viewpoint of MHD. This approach turns out to be ever more powerful when applied to streaming plasmas (the vast majority of visible matter in the Universe), toroidal plasmas (the most promising approach to fusion energy), and nonlinear dynamics (where it all comes together with modern computational techniques and extreme transonic and relativistic plasma flows).

The textbook interweaves theory and explicit calculations of waves and instabilities of streaming plasmas in complex magnetic geometries. It is ideally suited to advanced undergraduate and graduate courses in plasma physics and astrophysics.

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With Applications to Laboratory and Astrophysical Plasmas

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To Antonia, 陆蓉 (Rong Lu), and Micheline

Preface

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Preface

This book, together with the preceding *Principles of Magnetohydrodynamics* (to be referred to as Volume [1]), describes the two main applications of plasma physics, laboratory research on thermonuclear fusion energy and plasma-astrophysics of the solar system, stars, accretion disks, etc., from the single viewpoint of magnetohydrodynamics (MHD). This provides effective methods and insights for the interpretation of plasma phenomena on virtually all scales, ranging from the laboratory to the Universe. The key issue is understanding the complexities of plasma dynamics in extended magnetic structures. In Volume [1], the classical MHD model was developed in great detail without omitting steps in the derivations. This necessitated restriction to ideal dissipationless plasmas, in static equilibrium and with inhomogeneity in one direction. In the present volume on Advanced Magnetohydrodynamics [2], these restrictions are relaxed one by one: introducing stationary background flows, resistivity and reconnection, two-dimensional toroidal geometry, linear and nonlinear computational techniques and transonic flows and shocks. These topics transform the subject into a vital new area with many applications in laboratory, space and astrophysical plasmas.

The two volumes now consist of five parts:

- I Plasma physics preliminaries (Volume [1], Chapters 1-3),
- II Basic magnetohydrodynamics (Volume [1], Chapters 4-11),
- III Flow and dissipation (Volume [2], Chapters 12-15),
- IV Toroidal plasmas (Volume [2], Chapters 16–18),
- V Nonlinear dynamics (Volume [2], Chapters 19–21).

Inevitably, with the chosen distinction of topics for Volume [1] (mostly ideal linear phenomena described by self-adjoint linear operators) and topics for Volume [2] (mostly non-ideal, toroidal and nonlinear phenomena), the difference between "basic" and "advanced" levels of magnetohydrodynamics could not be strictly maintained. The logical order required inclusion of some advanced topics in Volume [1],

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whereas some topics that now appear in Volume [2] (like stationary flows and toroidal effects) really belong to the "principles" of MHD. Difficult parts or asides with tedious derivations, that may be skipped on first reading, are again indicated by a star (*) or put in small print in between triangles ($\triangleright \cdots \triangleleft$).

An overview of the subject matter of the different chapters of the two volumes may help the reader to find his way.

Contents of Volume [1]:

- Chapter 1 gives an introduction to laboratory fusion and astrophysical plasmas, and formulates provisional microscopic and macroscopic definitions of the plasma state.
- Chapter 2 discusses the three complementary points of view of single particle motion, kinetic theory, and fluid description. The corresponding theoretical models provide the opportunity to introduce some of the basic concepts of plasma physics.
- Chapter 3 gives the "derivation" of the macroscopic equations from the kinetic (Boltzmann) equation. Quotation marks because a fully satisfactory derivation can not be given at present in view of the largely unknown contribution of turbulent transport processes. The presentation given is meant to provide some idea on the limitations of the macroscopic view point.
- Chapter 4 defines the MHD model and introduces the concept of scale independence. The central importance of the conservation laws is discussed at length. Based on this, the similarities and differences of laboratory and astrophysical plasmas are articulated in terms of a number of generic boundary value problems.
- Chapter 5 derives the basic MHD waves and describes their properties, with an eye on their role in spectral analysis and computational MHD. The theory of characteristics is introduced as a way to describe the propagation of nonlinear disturbances.
- Chapter 6 treats the subject of waves and instabilities from the unifying point of view of spectral theory. The force operator formulation and the energy principle are extensively discussed. The analogy with quantum mechanics is pointed out and exploited. The difficult extension to interface systems is treated in detail.
- Chapter 7 applies the spectral analysis developed in Chapter 6 to inhomogeneous plasmas in a plane slab. The wave equation for gravito-MHD waves is derived and solved in various limits. Here, all the intricacies of the subject enter: continuous spectra, damping of Alfvén waves, local instabilities, etc. The analogy between helioseismology and MHD spectroscopy in tokamaks is shown to hold great promise for the investigation of plasma dynamics.
- Chapter 8 introduces the enormous variety of magnetic phenomena in astrophysics, in particular for the solar system (dynamo, solar wind, magnetospheres, etc.), and provides basic examples of plasma dynamics worked out in later chapters.
- Chapter 9 is the cylindrical counterpart of Chapter 7, with a wave equation describing the various waves and instabilities. It presents the stability analysis of diffuse cylindrical plasmas (classical pinches and present tokamaks) from the spectral perspective.

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- Chapter 10 solves the initial value problem for one-dimensional inhomogeneous MHD and the associated damping due to the continuous spectrum.
- Chapter 11 discusses resonant absorption and phase mixing in the context of heating mechanisms of solar and stellar coronae. Sunspot seismology is introduced as another example of MHD spectroscopy.

Contents of Volume [2]:

- Chapter 12 initiates the most urgent extension of the theory presented in Volume [1]: waves and instabilities in plasmas with stationary background flows, a theme of common interest for laboratory fusion and astrophysical plasma research. The old problem of how to find the complex eigenvalues of stationary plasmas is solved by means of a new method of constructing solution paths in the complex plane.
- Chapter 13 applies the new theory of Chapter 12 to the two classical topics of shear flow in plane plasma slabs, including the Kelvin–Helmholtz instability, and to rotation in cylindrical plasmas, including the magneto-rotational instability.
- Chapter 14 treats the considerable modification of plasma dynamics when resistivity is introduced in the MHD description, both in the linear domain of spectral theory and in the nonlinear domain of reconnection.
- Chapter 15 introduces the basic techniques of computational MHD, the discretization techniques, the methods of time stepping, etc. It thus provides the modern techniques needed to solve for the dynamics of plasmas in complicated magnetic geometries.
- Chapter 16 presents the classical theory of static equilibrium of toroidal plasmas, a topic of central interest in fusion research of tokamaks. Both analytical theory and numerical solutions are presented.
- Chapter 17 concerns the spectral theory of waves and instabilities in toroidal equilibria, again a central topic in tokamak research. Because of this important application, this part of MHD spectral theory is the most developed one, also with respect to comparison with experimental data. This activity is rightly called *MHD spectroscopy*.
- Chapter 18 introduces the theory of transonic equilibria and spectral theory of those equilibria, a subject of huge interest, but still in its infancy.
- Chapter 19 presents the counterpart of Chapter 15 by introducing the numerical methods for nonlinear MHD, in particular for plasmas with large background flows, applied in the last two chapters.
- Chapter 20 discusses the MHD shock conditions from a new perspective, scale independence leading to time reversal duality, and it introduces some of the important areas of application of nonlinear MHD, viz. astrophysical winds and transonic flows.
- Chapter 21 introduces special relativistic MHD, in particular the linear waves and nonlinear shocks that occur at relativistic speeds. The books ends with applications to astrophysical phenomena, like relativistic jets, and thus completes the panorama of the tremendously exciting field of magnetohydrodynamics dominated by flows.

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It is impossible to include all topics that actually belong to the field of advanced MHD. Fortunately, books or chapters of books exist on most of those topics, like *dynamos* (Moffatt [337], Ortolani & Schnack [357], Ferriz-Mas & Núñez [133], Rüdiger & Hollerbach [397]); *chaos* (White [483]); *stellarators* (Freidberg [141]); *spheromaks* (Bellan [31]); *anomalous transport* (Balescu [21], Yoshizawa, Itoh & Itoh [492]); *MHD turbulence* (Biskamp [47]).

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Circle theorem: The actual time to complete a project is precisely π times the best estimate of the time that one foresees at the beginning of it.

Proof: Standing at the disk of the unknown, the best estimate is based on how long it takes to reach the other side, the actual time spent involves encircling it so as to really enclose it from all sides. That path is precisely π times longer; QED.

Finally, a frequently asked question is: "Will there be a third volume?" Yes, there will be, and you, the serious students of these two volumes who realized that these are just introductions to an enormous field of largely unexplored territory, are going to write it. Remember, with plasmas making up 90% of all (so far visible) matter of the Universe, and plasma physics under-represented in the physics curriculum of the universities, there is no doubt that there will be completely unexpected discoveries for you in store. Nature is on your side!