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Excerpt
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Chapter I
Introduction

In dealing with partial differential equations, it is useful to differentiate between several types. In particular, we classify partial differential equations of second order as *elliptic, hyperbolic*, and *parabolic*. Both the theoretical and numerical treatment differ considerably for the three types. For example, in contrast with the case of ordinary differential equations where either initial or boundary conditions can be specified, here the type of equation determines whether initial, boundary, or initial-boundary conditions should be imposed.

The most important application of the finite element method is to the numerical solution of elliptic partial differential equations. Nevertheless, it is important to understand the differences between the three types of equations. In addition, we present some elementary properties of the various types of equations. Our discussion will show that for differential equations of elliptic type, we need to specify boundary conditions and not initial conditions.

There are two main approaches to the numerical solution of elliptic problems: *finite difference methods* and *variational methods*. The finite element method belongs to the second category. Although finite element methods are particularly effective for problems with complicated geometry, finite difference methods are often employed for simple problems, primarily because they are simpler to use. We include a short and elementary discussion of them in this chapter.

§ 1. Examples and Classification of PDE's

Examples

We first consider some examples of second order partial differential equations which occur frequently in physics and engineering, and which provide the basic prototypes for elliptic, hyperbolic, and parabolic equations.

1.1 Potential Equation. Let Ω be a domain in $\mathbb{R}^2$. Find a function u on Ω with

$$u_{xx} + u_{yy} = 0. \tag{1.1}$$

This is a differential equation of second order. To determine a unique solution, we must also specify boundary conditions.

One way to get solutions of (1.1) is to identify $\mathbb{R}^2$ with the complex plane. It is known from function theory that if $w(z) = u(z) + iv(z)$ is a holomorphic function on Ω, then its real part u and imaginary part v satisfy the potential equation. Moreover, u and v are infinitely often differentiable in the interior of Ω, and attain their maximum and minimum values on the boundary.

For the case where $\Omega := \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$ is a disk, there is a simple formula for the solution. Since $z^k = (re^{i\phi})^k$ is holomorphic, it follows that

$$r^k \cos k\phi, \quad r^k \sin k\phi, \quad \text{for } k = 0, 1, 2, \ldots,$$

satisfy the potential equation. If we expand these functions on the boundary in Fourier series,

$$u(\cos\phi, \sin\phi) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\phi + b_k \sin k\phi),$$

we can represent the solution in the interior as

$$u(x, y) = a_0 + \sum_{k=1}^{\infty} r^k (a_k \cos k\phi + b_k \sin k\phi). \tag{1.2}$$

The differential operator in (1.1) is the *two-dimensional Laplace operator*. For functions of d variables, it is

$$\Delta u := \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}.$$

The potential equation is a special case of the Poisson equation.

1.2 Poisson Equation. Let Ω be a domain in $\mathbb{R}^d$, $d = 2$ or 3. Here $f : \Omega \to \mathbb{R}$ is a prescribed charge density in Ω, and the solution u of the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega \tag{1.3}$$

describes the potential throughout Ω. As with the potential equation, this type of problem should be posed with boundary conditions.

1.3 The Plateau Problem as a Prototype of a Variational Problem. Suppose we stretch an ideal elastic membrane over a wire frame to create a drum. Suppose the wire frame is described by a closed, rectifiable curve in $\mathbb{R}^3$, and suppose that its parallel projection onto the (x, y)-plane is a curve with no double points. Then the position of the membrane can be described as the graph of a function $u(x, y)$. Because of the elasticity, it must assume a position such that its surface area

$$\int_\Omega \sqrt{1 + u_x^2 + u_y^2}\, dxdy$$

is minimal.

In order to solve this nonlinear variational problem approximately, we introduce a simplification. Since $\sqrt{1+z} = 1 + \frac{z}{2} + \mathcal{O}(z^2)$, for small values of u_x and u_y we can replace the integrand by a quadratic expression. This leads to the problem

$$\frac{1}{2}\int_\Omega (u_x^2 + u_y^2)\, dxdy \to \min! \tag{1.4}$$

The values of u on the boundary $\partial\Omega$ are prescribed by the given curve. We now show that the minimum is characterized by the associated Euler equation

$$\Delta u = 0. \tag{1.5}$$

Since such variational problems will be dealt with in more detail in Chapter II, here we establish (1.5) only on the assumption that a minimal solution u exists in $C^2(\Omega) \cap C^0(\bar{\Omega})$. If a solution belongs to $C^2(\Omega) \cap C^0(\bar{\Omega})$, it is called a *classical solution*. Let

$$D(u, v) := \int_\Omega (u_x v_x + u_y v_y)\, dxdy$$

and $D(v) := D(v, v)$. The quadratic form D satisfies the binomial formula

$$D(u + \alpha v) = D(u) + 2\alpha D(u, v) + \alpha^2 D(v).$$

Let $v \in C^1(\Omega)$ and $v|_{\partial\Omega} = 0$. Since $u + \alpha v$ for $\alpha \in \mathbb{R}$ is an admissible function for the minimum problem (1.4), we have $\frac{\partial}{\partial\alpha} D(u + \alpha v) = 0$ for $\alpha = 0$. Using the above binomial formula, we get $D(u, v) = 0$. Now applying Green's integral formula, we have

$$\begin{aligned} 0 = D(u, v) &= \int_\Omega (u_x v_x + u_y v_y)\, dxdy \\ &= -\int_\Omega v(u_{xx} + u_{yy})\, dxdy + \int_{\partial\Omega} v(u_x dy - u_y\, dx). \end{aligned}$$

The contour integral vanishes because of the boundary condition for v. The first integral vanishes for all $v \in C^1(\Omega)$ if and only if $\Delta u = u_{xx} + u_{yy} = 0$. This proves that (1.5) characterizes the solution of the (linearized) Plateau problem.

1.4 The Wave Equation as a Prototype of a Hyperbolic Differential Equation. The motion of particles in an ideal gas is subject to the following three laws, where as usual, we denote the velocity by v, the density by ρ, and the pressure by p:

1. *Continuity Equation.*

$$\frac{\partial \rho}{\partial t} = -\rho_0 \operatorname{div} v.$$

   Because of conservation of mass, the change in the mass contained in a (partial) volume V must be equal to the flow through its surface, i.e., it must be equal to $\int_{\partial V} \rho v \cdot n\, dO$. Applying Gauss' integral theorem, we get the above equation. Here ρ is approximated by the fixed density ρ_0.

2. *Newton's Law.*

$$\rho_0 \frac{\partial v}{\partial t} = -\operatorname{grad}\ p.$$

   The gradient in pressure induces a force field which causes the acceleration of the particles.

3. *State Equation.*

$$p = c^2 \rho.$$

   In ideal gases, the pressure is proportional to the density for constant temperature.

Using these three laws, we conclude that

$$\begin{aligned} \frac{\partial^2}{\partial t^2} p = c^2 \frac{\partial^2 \rho}{\partial t^2} &= -c^2 \frac{\partial}{\partial t} \rho_0 \operatorname{div} v = -c^2 \operatorname{div}(\rho_0 \frac{\partial v}{\partial t}) \\ &= c^2 \operatorname{div} \operatorname{grad} p = c^2 \Delta p. \end{aligned}$$

Other examples of the *wave equation*

$$u_{tt} = c^2 \Delta u$$

arise in two space dimensions for vibrating membranes, and in the one-dimensional case for a vibrating string. In one space dimension, the equation simplifies when c is normalized to 1:

$$u_{tt} = u_{xx}. \tag{1.6}$$

The wave equation leads to a well-posed problem (see Definition 1.8 below) when combined with initial conditions of the form

$$\begin{aligned} u(x,0) &= f(x), \\ u_t(x,0) &= g(x). \end{aligned} \tag{1.7}$$

1.5 Solution of the One-dimensional Wave Equation. To solve the wave equation (1.6)–(1.7), we apply the transformation of variables

$$\begin{aligned} \xi &= x + t, \\ \eta &= x - t. \end{aligned} \tag{1.8}$$

Applying the chain rule $u_x = u_\xi \frac{\partial \xi}{\partial x} + u_\eta \frac{\partial \eta}{\partial x}$, etc., we easily get

$$\begin{aligned} u_x &= u_\xi + u_\eta, & u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \\ u_t &= u_\xi - u_\eta, & u_{tt} &= u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}. \end{aligned} \tag{1.9}$$

Substituting the formulas (1.9) in (1.6) gives

$$4u_{\xi\eta} = 0.$$

The general solution is

$$\begin{aligned} u &= \phi(\xi) + \psi(\eta) \\ &= \phi(x+t) + \psi(x-t), \end{aligned} \tag{1.10}$$

where ϕ and ψ are functions which can be determined from the initial conditions (1.7):

$$\begin{aligned} f(x) &= \phi(x) + \psi(x), \\ g(x) &= \phi'(x) - \psi'(x). \end{aligned}$$

After differentiating the first equation, we have two equations for ϕ' and ψ' which are easily solved:

$$\begin{aligned} \phi' &= \frac{1}{2}(f' + g), & \phi(\xi) &= \frac{1}{2} f(\xi) + \frac{1}{2}\int_{x_0}^{\xi} g(s)\,ds, \\ \psi' &= \frac{1}{2}(f' - g), & \psi(\eta) &= \frac{1}{2} f(\eta) - \frac{1}{2}\int_{x_0}^{\eta} g(s)\,ds. \end{aligned}$$

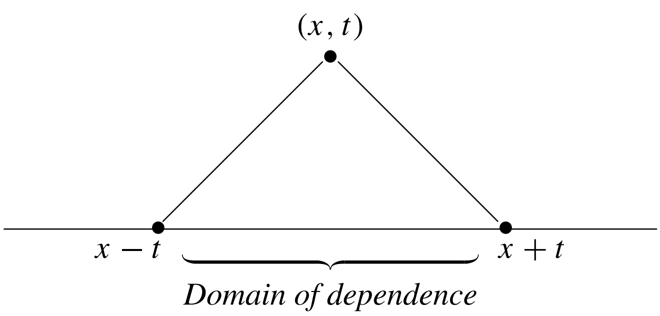

Fig. 1. Domain of dependence for the wave equation

Finally, using (1.10) we get

$$u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds. \tag{1.11}$$

We emphasize that the solution $u(x,t)$ depends only on the initial values between the points $x-t$ and $x+t$ (see Fig. 1). [If the constant c is not normalized to be 1, the dependence is on all points between $x-ct$ and $x+ct$]. This corresponds to the fact that in the underlying physical system, any change of data can only propagate with a finite velocity.

The solution u in (1.11) was derived on the assumption that it is twice differentiable. If the initial functions f and g are not differentiable, then neither are ϕ, ψ and u. However, the formula (1.11) remains correct and makes sense even in the nondifferentiable case.

1.6 The Heat Equation as a Prototype of a Parabolic Equation. Let $T(x,t)$ be the distribution of temperature in an object. Then the heat flow is given by

$$F = -\kappa \operatorname{grad} T,$$

where κ is the diffusion constant which depends on the material. Because of conservation of energy, the change in energy in a volume element is the sum of the heat flow through the surface and the amount of heat injection Q. Using the same arguments as for conservation of mass in Example 1.4, we have

$$\begin{aligned}\frac{\partial E}{\partial t} &= -\operatorname{div}\ F + Q\\ &= \operatorname{div}\ \kappa \operatorname{grad}\ T + Q\\ &= \kappa\Delta T + Q,\end{aligned}$$

where κ is assumed to be constant. Introducing the constant $a = \partial E/\partial T$ for the specific heat (which also depends on the material), we get

$$\frac{\partial T}{\partial t} = \frac{\kappa}{a}\Delta T + \frac{1}{a}Q.$$

For a one-dimensional rod and $Q = 0$, with $u = T$ this simplifies to

$$u_t = \sigma u_{xx}, \tag{1.12}$$

where $\sigma = \kappa/a$. As before, we may assume the normalization $\sigma = 1$ by an appropriate choice of units.

Parabolic problems typically lead to *initial-boundary-value problems.*

We first consider the heat distribution on a rod of finite length ℓ. Then, in addition to the initial values, we also have to specify the temperature or the heat fluxes on the boundaries. For simplicity, we restrict ourselves to the case where the temperature is constant at both ends of the rod as a function of time. Then, without loss of generality, we can assume that

$$\sigma = 1, \quad \ell = \pi \quad \text{and} \quad u(0, t) = u(\pi, t) = 0;$$

cf. Problem 1.10. Suppose the initial values are given by the Fourier series expansion

$$u(x, 0) = \sum_{k=1}^{\infty} a_k \sin kx, \qquad 0 < x < \pi.$$

Obviously, the functions $e^{-k^2 t} \sin kx$ satisfy the heat equation $u_t = u_{xx}$, and thus

$$u(x, t) = \sum_{k=1}^{\infty} a_k e^{-k^2 t} \sin kx, \qquad t \geq 0 \tag{1.13}$$

is a solution of the given initial-value problem.

For an infinitely long rod, the boundary conditions drop out. Now we need to know something about the decay of the initial values at infinity, which we ignore here. In this case we can write the solution using Fourier integrals instead of Fourier series. This leads to the representation

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\xi^2/4t} f(x - \xi)\, d\xi, \tag{1.14}$$

where the initial value $f(x) := u(x, 0)$ appears explicitly. Note that the solution at a point (x, t) depends on the initial values on the entire domain, and the propagation of the data occurs with infinite speed.

Classification of PDE's

Problems involving ordinary differential equations can be posed with either initial or boundary conditions. This is no longer the case for partial differential equations. Here the question of whether initial or boundary conditions should be applied depends on the *type of the differential equation*.

The general linear partial differential equation of second order in n variables $x = (x_1, \dots, x_n)$ has the form

$$-\sum_{i,k=1}^{n} a_{ik}(x)u_{x_i x_k} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u = f(x). \tag{1.15}$$

If the functions a_{ik}, b_i and c do not depend on x, then the partial differential equation has *constant coefficients*. Since $u_{x_i x_k} = u_{x_k x_i}$ for any function which is twice continuously differentiable, without loss of generality we can assume the symmetry $a_{ik}(x) = a_{ki}(x)$. Then the corresponding $n \times n$ matrix

$$A(x) := (a_{ik}(x))$$

is symmetric.

1.7 Definition. (1) The equation (1.15) is called *elliptic at the point* x provided $A(x)$ is positive definite.
(2) The equation (1.15) is called *hyperbolic at the point* x provided $A(x)$ has one negative and $n-1$ positive eigenvalues.
(3) The equation (1.15) is called *parabolic at the point* x provided $A(x)$ is positive semidefinite, but is not positive definite, and the rank of $(A(x), b(x))$ equals n.
(4) An equation is called *elliptic, hyperbolic* or *parabolic* provided it has the corresponding property for all points of the domain. □

In the elliptic case, the equation (1.15) is usually written in the compact form

$$Lu = f, \tag{1.16}$$

where L is an *elliptic differential operator of order* 2. The part with the derivatives of highest order, i.e., $-\sum a_{ik}(x)u_{x_i x_k}$, is called the *principal part* of L. For hyperbolic and parabolic problems there is a special variable which is usually time. Thus, hyperbolic differential equations can often be written in the form

$$u_{tt} + Lu = f, \tag{1.17}$$

while parabolic ones can often be written in the form

$$u_t + Lu = f, \tag{1.18}$$

where L is an elliptic differential operator.

If a differential equation is invariant under *isometric mappings* (i.e., under translation and rotation), then the elliptic operator has the form

$$Lu = -a_0 \Delta u + c_0 u.$$

The above examples all display this invariance.

Well-posed Problems

What happens if we consider a partial differential equation in a framework which is meant for a different type?

To answer this question, we first turn to the wave equation (1.6), and attempt to solve the *boundary-value problem* in the domain

$$\Omega = \{(x,t) \in \mathbb{R}^2;\ a_1 < x+t < a_2,\ b_1 < x-t < b_2\}.$$

Here Ω is a rotated rectangle, and its sides are parallel to the coordinate axes ξ, η defined in (1.8). In view of $u(\xi,\eta) = \phi(\xi) + \psi(\eta)$, the values of u on opposite sides of Ω can differ only by a constant. Thus, the boundary-value problem with general data is not solvable. This also follows for differently shaped domains by similar but somewhat more involved considerations.

Next we study the potential equation (1.1) in the domain $\{(x,y) \in \mathbb{R}^2;\ y \ge 0\}$ as an *initial-value problem*, where y plays the role of time. Let $n > 0$. Assuming

$$u(x,0) = \frac{1}{n}\sin nx,$$
$$u_y(x,0) = 0,$$

we clearly get the *formal solution*

$$u(x,y) = \frac{1}{n}\cosh ny\ \sin nx,$$

which grows like e^{ny}. Since n can be arbitrarily large, we draw the following conclusion: there exist *arbitrarily small initial values* for which the corresponding solution at $y = 1$ is *arbitrarily large*. This means that solutions of this problem, when they exist, are not stable with respect to perturbations of the initial values.

Using the same arguments, it is immediately clear from (1.13) that a solution of a parabolic equation is well-behaved for $t > t_0$, but not for $t < t_0$. However, sometimes we want to solve the heat equation in the backwards direction, e.g., in order to find out what initial temperature distribution is needed in order to get a prescribed distribution at a later time $t_1 > 0$. This is a well-known improperly posed problem. By (1.13), we can prescribe at most the low frequency terms of the temperature at time t_1, but by no means the high frequency ones.

Considerations of this type led Hadamard [1932] to consider the solvability of differential equations (and similarly structured problems) together with the stability of the solution.

1.8 Definition. A problem is called *well posed* provided it has a unique solution which depends continuously on the given data. Otherwise it is called *improperly posed*.

In principle, the question of whether a problem is well posed can depend on the choice of the norm used for the corresponding function spaces. For example, from (1.11) we see that problem (1.6)–(1.7) is well posed. The mapping

$$\begin{aligned} C(\mathbb{R}) \times C(\mathbb{R}) &\longrightarrow C(\mathbb{R} \times \mathbb{R}_+), \\ f, g &\longmapsto u \end{aligned}$$

defined by (1.11) is continuous provided $C(\mathbb{R})$ is endowed with the usual maximum norm, and $C(\mathbb{R} \times \mathbb{R}_+)$ is endowed with the weighted norm

$$\|u\| := \max_{x,t} \{\frac{|u(x,t)|}{1+|t|}\}.$$

The maximum principle to be discussed in the next section is a useful tool for showing that elliptic and parabolic differential equations are well posed.

Problems

1.9 Consider the potential equation in the disk $\Omega := \{(x,y) \in \mathbb{R}^2;\ x^2+y^2 < 1\}$, with the boundary condition

$$\frac{\partial}{\partial r} u(x) = g(x) \quad \text{for } x \in \partial\Omega$$

on the derivative in the normal direction. Find the solution when g is given by the Fourier series

$$g(\cos\phi, \sin\phi) = \sum_{k=1}^{\infty} (a_k \cos k\phi + b_k \sin k\phi)$$

without a constant term. (The reason for the lack of a constant term will be explained in Ch. II, §3.)

1.10 Consider the heat equation (1.12) for a rod with $\sigma \neq 1$, $\ell \neq \pi$ and $u(0,t) = u(\ell,t) = T_0 \neq 0$. How should the scalars, i.e., the constants in the transformations $t \longmapsto \alpha t$, $x \longmapsto \beta x$, $u \longmapsto u+\gamma$, be chosen so that the problem reduces to the normalized one?