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Excerpt
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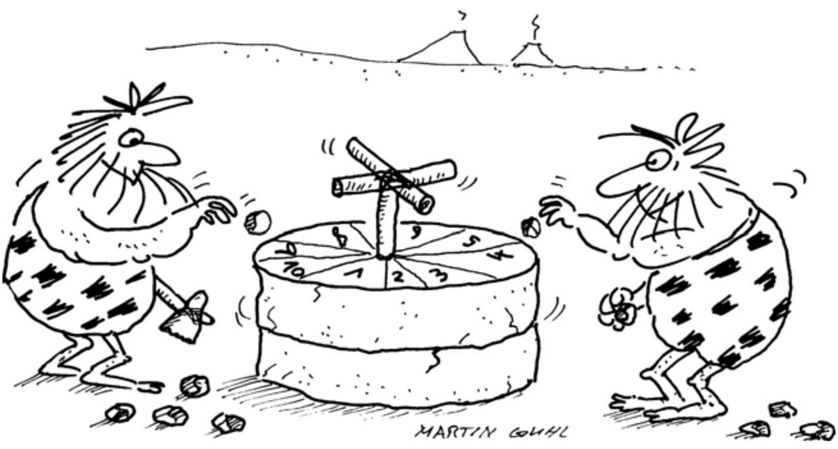
Introduction

It is difficult to say who had a greater impact on the mobility of goods in the preindustrial economy: the inventor of the wheel or the crafter of the first pair of dice. One thing, however, is certain: the genius that designed the first random-number generator, like the inventor of the wheel, will very likely remain anonymous forever. We do know that the first dice-like exemplars were made a very long time ago. Excavations in the Middle East and in India reveal that dice were already in use at least fourteen centuries before Christ. Earlier still, around 3500 B.C., a board game existed in Egypt in which players tossed four-sided sheep bones. Known as the *astragalus*, this precursor to the modern-day die remained in use right up to the Middle Ages.

In the sixteenth century, the game of dice, or craps as we might call it today, was subjected for the first time to a formal mathematical study by the Italian mathematician and physician Gerolamo Cardano (1501–1576). An ardent gambler, Cardano wrote a handbook for gamblers entitled *Liber de Ludo Aleae* (The



Book of Games of Chance) about probabilities in games of chance. Cardano originated and introduced the concept of the set of outcomes of an experiment, and for cases in which all outcomes are equally probable, he defined the probability of any one event occurring as the ratio of the number of favorable outcomes and the total number of possible outcomes. This may seem obvious today, but in Cardano's day such an approach marked an enormous leap forward in the development of probability theory. This approach, along with a correct counting of the number of possible outcomes, gave the famous astronomer and physicist Galileo Galilei the tools he needed to explain to the Grand Duke of Tuscany, his benefactor, why it is that when you toss three dice, the chance of the sum being 10 is greater than the chance of the sum being 9 (the probabilities are $\frac{27}{216}$ and $\frac{25}{216}$, respectively).

By the end of the seventeenth century, the Dutch astronomer Christiaan Huygens (1625–1695) laid the foundation for current probability theory. His text *Van Rekeningh in Spelen van Geluck* (On Reasoning in Games of Chance), published in 1660, had enormous influence on later developments in probability theory (this text had already been translated into Latin under the title *De Ratiociniis de Ludo Aleae* in 1657). It was Huygens who originally introduced the concept of expected value, which plays such an important role in probability theory. His work unified various problems that had been solved earlier by the famous French mathematicians Pierre Fermat and Blaise Pascal. Among these was the interesting problem of how two players in a game of chance should divide the stakes if the game ends prematurely. Huygens' work led the field for many years until, in 1713, the Swiss mathematician Jakob Bernoulli (1654–1705) published *Ars Conjectandi* (The Art of Conjecturing) in which he presented the first general theory for calculating probabilities. Then, in 1812, the great French mathematician Pierre Simon Laplace (1749–1827) published his *Théorie Analytique des Probabilités*. This book unquestionably represents the greatest contribution in the history of probability theory.

Fermat and Pascal established the basic principles of probability in their brief correspondence during the summer of 1654, in which they considered some of the specific problems of odds calculation that had been posed to them by gambling acquaintances. One of the more well known of these problems is that of the Chevalier de Méré, who claimed to have discovered a contradiction in arithmetic. De Méré knew that it was advantageous to wager that a six would be rolled at least one time in four rolls of one die, but his experience as gambler taught him that it was not advantageous to wager on a double six being rolled at least one time in 24 rolls of a pair of dice. He argued that there were six possible outcomes for the toss of a single die and 36 possible outcomes for the toss of a pair of dice, and he claimed that this evidenced a contradiction to the arithmetic law of proportions, which says that the ratio of 4 to 6 should be the same as 24 to 36. De Méré turned to Pascal, who showed him with a few simple calculations that probability does not follow the law of proportions, as De Méré had mistakenly assumed (by De Méré's logic, the probability of at least one head in two tosses of a fair coin would be $2 \times 0.5 = 1$, which we know cannot be true). In any case, De Méré must have been an ardent player in order to have established empirically that the probability of rolling at least one double six in 24 rolls of a pair of dice lies just under one-half. The precise value of this probability is 0.4914. The probability of rolling at least one six in four rolls of a single die can be calculated as 0.5177. Incidentally, you may find it surprising that four rolls of a die are required, rather than three, in order to have about an equal chance of rolling at least one six.

Modern probability theory

Although probability theory was initially the product of questions posed by gamblers about their odds in the various games of chance, in its modern form, it has far outgrown any boundaries associated with the gaming room. These days, probability theory plays an increasingly greater roll in many fields. Countless problems in our daily lives call for a probabilistic approach. In many cases, better judicial and medical decisions result from an elementary knowledge of probability theory. It is essential to the field of insurance.† And likewise, the stock market, "the largest casino in the world," cannot do without it. The telephone network with its randomly fluctuating load could not have been economically designed without the aid of probability theory. Call-centers and airline companies apply probability theory to determine how many telephone lines and service desks will be needed based on expected demand. Probability theory is also essential in stock control to find a balance between the stock-out probability and the costs of holding inventories in an environment of uncertain demand. Engineers use probability theory when constructing dikes to calculate the probability of water levels exceeding their margins; this gives them the information they need to determine optimum dike elevation. These examples underline the extent to which the theory of probability has become an integral part of our lives. Laplace was right when he wrote almost 200 years ago in his Théorie Analytique des Probabilités:

> The theory of probabilities is at bottom nothing but common sense reduced to calculus; it enables us to appreciate with exactness that which accurate minds feel

† Actuarial scientists have been contributing to the development of probability theory since its early stages. Also, astronomers have played very important roles in the development of probability theory.

> with a sort of instinct for which ofttimes they are unable to account. . . . It teaches us to avoid the illusions which often mislead us; . . . there is no science more worthy of our contemplations nor a more useful one for admission to our system of public education.

Probability theory and simulation

In terms of practical range, probability theory is comparable with geometry; both are branches of applied mathematics that are directly linked with the problems of daily life. But while pretty much anyone can call up a natural feel for geometry to some extent, many people clearly have trouble with the development of a good intuition for probability. Probability and intuition do not always agree. In no other branch of mathematics is it so easy to make mistakes as in probability theory. The development of the foundations of probability theory took a long time and went accompanied with ups and downs. The reader facing difficulties in grasping the concepts of probability theory might find comfort in the idea that even the genius Gottfried von Leibniz (1646–1716), the inventor of differential and integral calculus along with Newton, had difficulties in calculating the probability of throwing 11 with one throw of two dice. Probability theory is a difficult subject to get a good grasp of, especially in a formal framework. The computer offers excellent possibilities for acquiring a better understanding of the basic ideas of probability theory by means of simulation. With computer simulation, a concrete probability situation can be imitated on the computer. The simulated results can then be shown graphically on the screen. The graphic clarity offered by such a computer simulation makes it an especially suitable means to acquiring a better feel for probability. Not only a didactic aid, computer simulation is also a practical tool for tackling probability problems that are too complicated for scientific solution. Computer simulation, for example, has made it possible to develop winning strategies in the game of blackjack.

An outline

Part One of the book comprises Chapters 1–6. These chapters introduce the reader to the basic concepts of probability theory by using motivating examples to illustrate the concepts. A "feel for probabilities" is first developed through examples that endeavor to bring out the essence of probability in a compelling way. Simulation is a perfect aid in this undertaking of providing insight into the hows and whys of probability theory. We will use computer simulation, when needed, to illustrate subtle issues. The two pillars of probability theory, namely, the *law of large numbers* and the *central limit theorem* receive in-depth treatment. The nature of these two laws is best illustrated through the coin-toss experiment. The law of large numbers says that the percentage of tosses to come out heads will be as close to 0.5 as you can imagine provided that the coin is tossed often enough. How often the coin must be tossed in order to reach a prespecified precision for the percentage can be identified with the central limit theorem.

In Chapter 1, readers first encounter a series of intriguing problems to test their feel for probabilities. These problems will all be solved in the ensuing chapters. In Chapter 2, the law of large numbers provides the central theme. This law makes a connection between the probability of an event in an experiment and the relative frequency with which this event will occur when the experiment is repeated a very large number of times. Formulated by the aforementioned Jakob Bernoulli, the law of large numbers forms the theoretical foundation under the experimental determination of probability by means of computer simulation. The law of large numbers is clearly illuminated by the repeated coin-toss experiment, which is discussed in detail in Chapter 2. Astonishing results hold true in this simple experiment, and these results blow holes in many a mythical assumption, such as the "hot hand" in basketball. One remarkable application of the law of large numbers can be seen in the Kelly formula, a betting formula that can provide insight for the making of horse racing and investment decisions alike. The basic principles of computer simulation will also be discussed in Chapter 2, with emphasis on the subject of how random numbers can be generated on the computer.

In Chapter 3, we will tackle a number of realistic probability problems. Each problem will undergo two treatments, the first one being based on computer simulation and the second bearing the marks of a theoretical approach. Lotteries and casino games are sources of inspiration for some of the problems in Chapter 3.

The binomial distribution, the Poisson distribution, and the hypergeometric distribution are the subjects of Chapter 4. We will discuss which of these important probability distributions applies to which probability situations, and we will take a look into the practical importance of the distributions. Once again, we look to the lotteries to provide us with instructional and entertaining examples. We will see, in particular, how important the sometimes underestimated Poisson distribution, named after the French mathematician Siméon-Denis Poisson (1781–1840), really is.

In Chapter 5, two more fundamental principles of probability theory and statistics will be introduced: the central limit theorem and the normal distribution with its bell-shaped probability curve. The central limit theorem is by far the most important product of probability theory. The names of the mathematicians Abraham de Moivre and Pierre Simon Laplace are inseparably linked to this theorem and to the normal distribution. De Moivre discovered the normal distribution around 1730.† An explanation of the frequent occurrence of this distribution is provided by the central limit theorem. This theorem states that data influenced by many small and unrelated random effects are approximately normally distributed. It has been empirically observed that various natural phenomena, such as the heights of individuals, intelligence scores, the luminosity of stars, and daily returns of the S&P, follow approximately a normal distribution. The normal curve is also indispensable in quantum theory in physics. It describes the statistical behavior of huge numbers of atoms or electrons. A great many statistical methods are based on the central limit theorem. For one thing, this theorem makes it possible for us to evaluate how (im)probable certain deviations from the expected value are. For example, is the claim that heads came up 5,250 times in 10,000 tosses of a fair coin credible? What are the margins of errors in the predictions of election polls? The standard deviation concept plays a key roll in the answering of these questions. We devote considerable attention to this fundamental concept, particularly in the context of investment issues. At the same time, we also demonstrate in Chapter 5, with the help of the central limit theorem, how confidence intervals for the outcomes of simulation studies can be constructed. The standard deviation concept also comes into play here. The central limit theorem will also be used to link the random walk model with the Brownian motion model. These models, which are used to describe the behavior of a randomly moving object, are among the most useful probability models in science. Applications in finance will be discussed, including the Black-Scholes formula for the pricing of options.

The probability tree concept is discussed in Chapter 6. For situations where the possibility of an uncertain outcome exists in successive phases, a probability tree can be made to systematically show what all of the possible paths are. Various applications of the probability tree concept will be considered, including the famous Monty Hall dilemma and the test paradox. In addition, we will also look at the Bayes formula in Chapter 6. This formula is a descriptive rule for revising probabilities in light of new information. Among other things, the Bayes rule is used in legal argumentation and in formulating medical diagnoses for specific illnesses. This eighteenth century formula, constructed by the English clergyman Thomas Bayes (1702–1761), laid the foundation for a separate branch of statistics, namely Bayesian statistics. Bayesian probability theory is historically

† The French-born Abraham de Moivre (1667–1754) lived most of his life in England. The protestant de Moivre left France in 1688 to escape religious persecution. He was a good friend of Isaac Newton and supported himself by calculating odds for gamblers and insurers and by giving private lessons to students.

the original approach to statistics, predating what is nowadays called classical statistics by a century. Astronomers have contributed much to Bayesian probability theory. In Bayesian probability one typically deals with nonrepeatable chance experiments. Astronomers cannot do experiments on the universe and thus have to make probabilistic inferences from evidence left behind. This is very much the same situation as in forensic science, in which Bayesian probability plays a very important role as well.

Part Two of the book is along the lines of a classical textbook and comprises Chapters 7–15. These chapters are intended for the more mathematically oriented reader. Chapter 7 goes more deeply into the axioms and rules of probability theory. In Chapter 8, the concept of conditional probability and the nature of Bayesian analysis are delved into more deeply. Properties of the expected value are discussed in Chapter 9. Chapter 10 gives an explanation of continuous distributions, always a difficult concept for the beginner to absorb, and provides insight into the most important probability densities. Whereas Chapter 10 deals with the probability distribution of a single random variable, Chapter 11 discusses joint probability distributions for two or more dependent random variables. The multivariate normal distribution is the most important joint probability distribution and is the subject of Chapter 12. Chapter 13 deals with conditional distributions and discusses the law of conditional expectations. In Chapter 14, we deal with the method of moment-generating functions. This powerful method enables us to analyze many applied probability problems. Also, the method is used to provide proofs for the strong law of large numbers and the central limit theorem. In the final Chapter 15, we introduce a random process, known as a Markov chain, which can be used to model many real-world systems that evolve dynamically in time in a random environment.

PART ONE

Probability in action

1

Probability questions

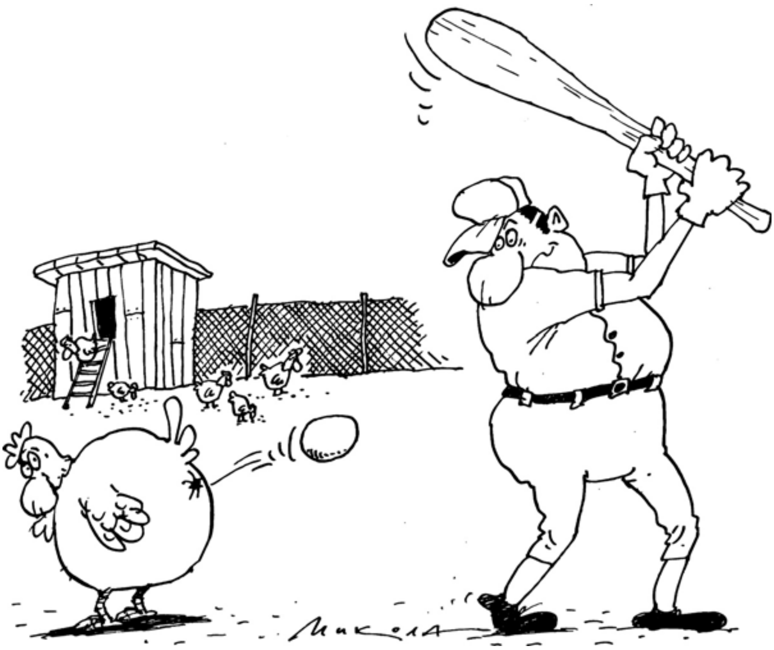
In this chapter, we provide a number of probability problems that challenge the reader to test his or her feeling for probabilities. As stated in the Introduction, it is possible to fall wide of the mark when using intuitive reasoning to calculate a probability, or to estimate the order of magnitude of a probability. To find out how you fare in this regard, it may be useful to try one or more of these 12 problems. They are playful in nature but are also illustrative of the surprises one can encounter in the solving of practical probability problems. Think carefully about each question before looking up its solution. All of the solutions to these problems can be found scattered throughout the ensuing chapters.

Question 1. A birthday problem (§3.1, §4.2.3)

You go with a friend to a football (soccer) game. The game involves 22 players of the two teams and one referee. Your friend wagers that, among these 23 persons on the field, at least two people will have birthdays on the same day. You will receive ten dollars from your friend if this is not the case. How much money should you, if the wager is to be a fair one, pay out to your friend if he is right?

Question 2. Probability of winning streaks (§2.1.3, §5.9.1)

A basketball player has a 50% success rate in free throw shots. Assuming that the outcomes of all free throws are independent from one another, what is the probability that, within a sequence of 20 shots, the player can score five baskets in a row?



Question 3. A scratch-and-win lottery (§4.2.3)

A scratch-and-win lottery dispenses 10,000 lottery tickets per week in Andorra and ten million in Spain. In both countries, demand exceeds supply. There are two numbers, composed of multiple digits, on every lottery ticket. One of these numbers is visible, and the other is covered by a layer of silver paint. The numbers on the 10,000 Andorran tickets are composed of four digits and the numbers on the ten million Spanish tickets are composed of seven digits. These numbers are randomly distributed over the quantity of lottery tickets, but in such a way that no two tickets display the same open or the same hidden number. The ticket holder wins a large cash prize if the number under the silver paint is revealed to be the same as the unpainted number on the ticket. Do you think the probability of at least one winner in the Andorran Lottery is significantly different from the probability of at least one winner in Spain? What is your estimate of the probability of a win occurring in each of the lotteries?

Question 4. A lotto problem (§4.2.3)

In each drawing of Lotto 6/45, six distinct numbers are drawn from the numbers 1, . . . , 45. In an analysis of 30 such lotto drawings, it was apparent that some