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Gauss's law for electric fields

In Maxwell's Equations, you'll encounter two kinds of electric field: the *electrostatic* field produced by electric charge and the *induced* electric field produced by a changing magnetic field. Gauss's law for electric fields deals with the electrostatic field, and you'll find this law to be a powerful tool because it relates the spatial behavior of the electrostatic field to the charge distribution that produces it.

1.1 The integral form of Gauss's law

There are many ways to express Gauss's law, and although notation differs among textbooks, the integral form is generally written like this:

$$\oint_S \vec{E} \circ \hat{n}\, da = \frac{q_{\text{enc}}}{\varepsilon_0} \quad \text{Gauss's law for electric fields (integral form).}$$

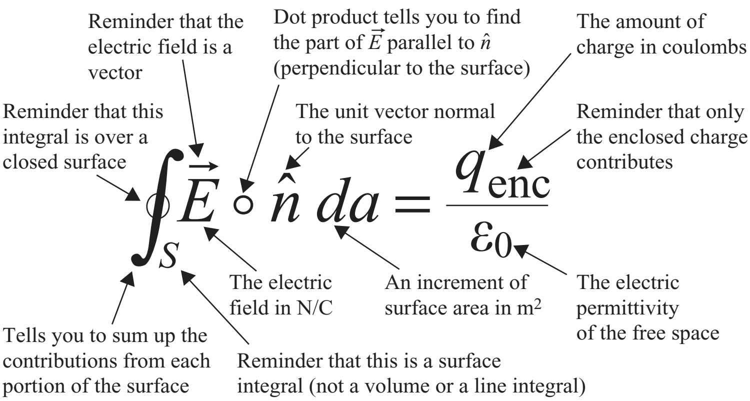
The left side of this equation is no more than a mathematical description of the electric flux – the number of electric field lines – passing through a closed surface S, whereas the right side is the total amount of charge contained within that surface divided by a constant called the permittivity of free space.

If you're not sure of the exact meaning of "field line" or "electric flux," don't worry – you can read about these concepts in detail later in this chapter. You'll also find several examples showing you how to use Gauss's law to solve problems involving the electrostatic field. For starters, make sure you grasp the main idea of Gauss's law:

> Electric charge produces an electric field, and the flux of that field passing through any closed surface is proportional to the total charge contained within that surface.

In other words, if you have a real or imaginary closed surface of any size and shape and there is no charge inside the surface, the electric flux through the surface must be zero. If you were to place some positive charge anywhere inside the surface, the electric flux through the surface would be positive. If you then added an equal amount of negative charge inside the surface (making the total enclosed charge zero), the flux would again be zero. Remember that it is the *net* charge enclosed by the surface that matters in Gauss's law.

To help you understand the meaning of each symbol in the integral form of Gauss's law for electric fields, here's an expanded view:



How is Gauss's law useful? There are two basic types of problems that you can solve using this equation:

(1) Given information about a distribution of electric charge, you can find the electric flux through a surface enclosing that charge.

(2) Given information about the electric flux through a closed surface, you can find the total electric charge enclosed by that surface.

The best thing about Gauss's law is that for certain highly symmetric distributions of charges, you can use it to find the electric field itself, rather than just the electric flux over a surface.

Although the integral form of Gauss's law may look complicated, it is completely understandable if you consider the terms one at a time. That's exactly what you'll find in the following sections, starting with $\vec{E}$, the electric field.

$\vec{E}$ The electric field

To understand Gauss's law, you first have to understand the concept of the electric field. In some physics and engineering books, no direct definition of the electric field is given; instead you'll find a statement that an electric field is "said to exist" in any region in which electrical forces act. But what exactly *is* an electric field?

This question has deep philosophical significance, but it is not easy to answer. It was Michael Faraday who first referred to an electric "field of force," and James Clerk Maxwell identified that field as the space around an electrified object – a space in which electric forces act.

The common thread running through most attempts to define the electric field is that fields and forces are closely related. So here's a very pragmatic definition: an electric field is the electrical force per unit charge exerted on a charged object. Although philosophers debate the true meaning of the electric field, you can solve many practical problems by thinking of the electric field at any location as the number of newtons of electrical force exerted on each coulomb of charge at that location. Thus, the electric field may be defined by the relation

$$\vec{E} \equiv \frac{\vec{F}_e}{q_0}, \tag{1.1}$$

where $\vec{F}_e$ is the electrical force on a small[1] charge q_0. This definition makes clear two important characteristics of the electric field:

(1) $\vec{E}$ is a vector quantity with magnitude directly proportional to force and with direction given by the direction of the force on a positive test charge.
(2) $\vec{E}$ has units of newtons per coulomb (N/C), which are the same as volts per meter (V/m), since volts = newtons × meters/coulombs.

In applying Gauss's law, it is often helpful to be able to visualize the electric field in the vicinity of a charged object. The most common approaches to constructing a visual representation of an electric field are to use a either arrows or "field lines" that point in the direction of the field at each point in space. In the arrow approach, the strength of the field is indicated by the length of the arrow, whereas in the field line

[1] Why do physicists and engineers always talk about small test charges? Because the job of this charge is to *test* the electric field at a location, not to add another electric field into the mix (although you can't stop it from doing so). Making the test charge infinitesimally small minimizes the effect of the test charge's own field.

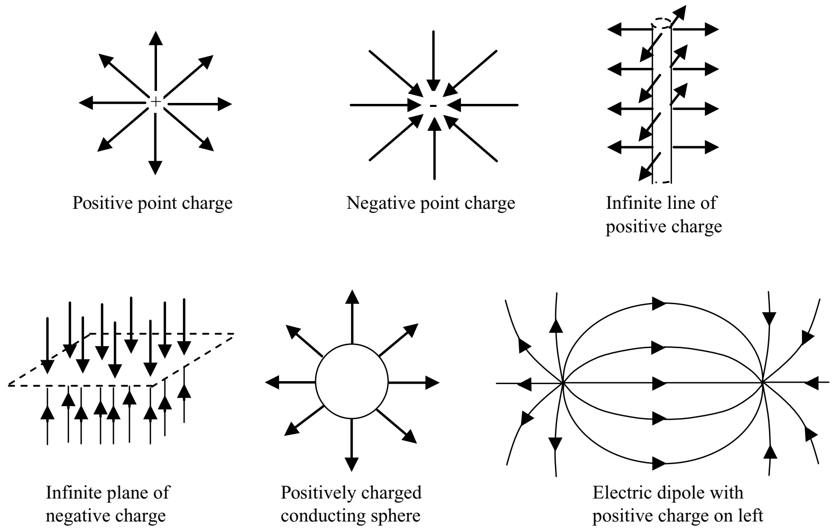


Figure 1.1 Examples of electric fields. Remember that these fields exist inthree dimensions; full three-dimensional (3-D) visualizations are available on the book's website.

approach, it is the spacing of the lines that tells you the field strength (with closer lines signifying a stronger field). When you look at a drawing of electric field lines or arrows, be sure to remember that the field exists between the lines as well.

Examples of several electric fields relevant to the application of Gauss's law are shown in Figure 1.1.

Here are a few rules of thumb that will help you visualize and sketch the electric fields produced by charges[2]:

- Electric field lines must originate on positive charge and terminate on negative charge.
- The net electric field at any point is the vector sum of all electric fields present at that point.
- Electric field lines can never cross, since that would indicate that the field points in two different directions at the same location (if two or more different sources contribute electric fields pointing in different directions at the same location, the total electric field is the vector sum

[2] In Chapter 3, you can read about electric fields produced not by charges but by changing magnetic fields. That type of field circulates back on itself and does not obey the same rules as electric fields produced by charge.

Table 1.1. *Electric field equations for simple objects*

Point charge (charge = q)	$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$ (at distance r from q)
Conducting sphere (charge = Q)	$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$ (outside, distance r from center) $\vec{E} = 0$ (inside)
Uniformly charged insulating sphere (charge = Q, radius = r_0)	$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$ (outside, distance r from center) $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Qr}{r_0^3}\hat{r}$ (inside, distance r from center)
Infinite line charge (linear charge density = λ)	$\vec{E} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}\hat{r}$ (distance r from line)
Infinite flat plane (surface charge density = σ)	$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$

of the individual fields, and the electric field lines always point in the single direction of the total field).

- Electric field lines are always perpendicular to the surface of a conductor in equilibrium.

Equations for the electric field in the vicinity of some simple objects may be found in Table 1.1.

So exactly what does the $\vec{E}$ in Gauss's law represent? It represents the total electric field at each point on the surface under consideration. The surface may be real or imaginary, as you'll see when you read about the meaning of the surface integral in Gauss's law. But first you should consider the dot product and unit normal that appear inside the integral.

○ The dot product

When you're dealing with an equation that contains a multiplication symbol (a circle or a cross), it is a good idea to examine the terms on both sides of that symbol. If they're printed in bold font or are wearing vector hats (as are $\vec{E}$ and $\hat{n}$ in Gauss's law), the equation involves vector multiplication, and there are several different ways to multiply vectors (quantities that have both magnitude and direction).

In Gauss's law, the circle between $\vec{E}$ and $\hat{n}$ represents the dot product (or "scalar product") between the electric field vector $\vec{E}$ and the unit normal vector $\hat{n}$ (discussed in the next section). If you know the Cartesian components of each vector, you can compute this as

$$\vec{A} \circ \vec{B} = A_x B_x + A_y B_y + A_z B_z. \tag{1.2}$$

Or, if you know the angle θ between the vectors, you can use

$$\vec{A} \circ \vec{B} = |\vec{A}||\vec{B}| \cos \theta, \tag{1.3}$$

where $|\vec{A}|$ and $|\vec{B}|$ represent the magnitude (length) of the vectors. Notice that the dot product between two vectors gives a *scalar* result.

To grasp the physical significance of the dot product, consider vectors $\vec{A}$ and $\vec{B}$ that differ in direction by angle θ, as shown in Figure 1.2(a).

For these vectors, the projection of $\vec{A}$ onto $\vec{B}$ is $|\vec{A}| \cos\theta$, as shown in Figure 1.2(b). Multiplying this projection by the length of $\vec{B}$ gives $|\vec{A}||\vec{B}| \cos\theta$. Thus, the dot product $\vec{A} \circ \vec{B}$ represents the projection of $\vec{A}$ onto the direction of $\vec{B}$ multiplied by the length of $\vec{B}$.[3] The usefulness of this operation in Gauss's law will become clear once you understand the meaning of the vector $\hat{n}$.

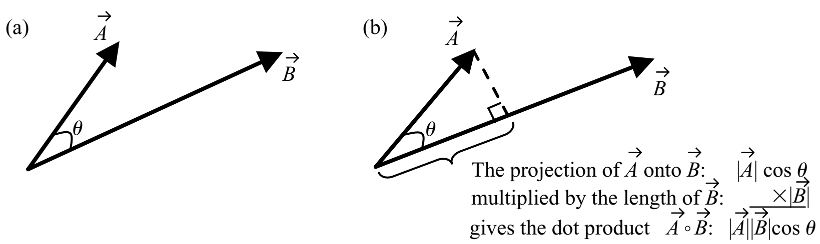


Figure 1.2 The meaning of the dot product.

[3] You could have obtained the same result by finding the projection of $\vec{B}$ onto the direction of $\vec{A}$ and then multiplying by the length of $\vec{A}$.

$\boxed{\hat{n}}$ The unit normal vector

The concept of the unit normal vector is straightforward; at any point on a surface, imagine a vector with length of one pointing in the direction perpendicular to the surface. Such a vector, labeled $\hat{n}$, is called a "unit" vector because its length is unity and "normal" because it is perpendicular to the surface. The unit normal for a planar surface is shown in Figure 1.3(a).

Certainly, you could have chosen the unit vector for the plane in Figure 1.3(a) to point in the opposite direction – there's no fundamental difference between one side of an open surface and the other (recall that an open surface is any surface for which it is possible to get from one side to the other without going *through* the surface).

For a closed surface (defined as a surface that divides space into an "inside" and an "outside"), the ambiguity in the direction of the unit normal has been resolved. By convention, the unit normal vector for a closed surface is taken to point outward – away from the volume enclosed by the surface. Some of the unit vectors for a sphere are shown in Figure 1.3(b); notice that the unit normal vectors at the Earth's North and South Pole would point in opposite directions if the Earth were a perfect sphere.

You should be aware that some authors use the notation $d\vec{a}$ rather than $\hat{n}\,da$. In that notation, the unit normal is incorporated into the vector area element $d\vec{a}$, which has magnitude equal to the area da and direction along the surface normal $\hat{n}$. Thus $d\vec{a}$ and $\hat{n}\,da$ serve the same purpose.

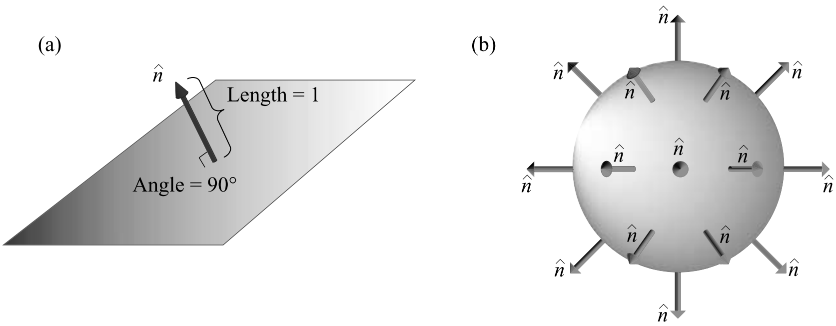


Figure 1.3 Unit normal vectors for planar and spherical surfaces.

$\vec{E} \circ \hat{n}$ The component of $\vec{E}$ normal to a surface

If you understand the dot product and unit normal vector, the meaning of $\vec{E} \circ \hat{n}$ should be clear; this expression represents the component of the electric field vector that is perpendicular to the surface under consideration.

If the reasoning behind this statement isn't apparent to you, recall that the dot product between two vectors such as $\vec{E}$ and $\hat{n}$ is simply the projection of the first onto the second multiplied by the length of the second. Recall also that by definition the length of the unit normal is one ($|\hat{n}| = 1$), so that

$$\vec{E} \circ \hat{n} = |\vec{E}||\hat{n}| \cos\theta = |\vec{E}| \cos\theta, \tag{1.4}$$

where θ is the angle between the unit normal $\hat{n}$ and $\vec{E}$. This is the component of the electric field vector perpendicular to the surface, as illustrated in Figure 1.4.

Thus, if $\theta = 90°$, $\vec{E}$ is perpendicular to $\hat{n}$, which means that the electric field is parallel to the surface, and $\vec{E} \circ \hat{n} = |\vec{E}| \cos(90°) = 0$. So in this case the component of $\vec{E}$ perpendicular to the surface is zero.

Conversely, if $\theta = 0°$, $\vec{E}$ is parallel to $\hat{n}$, meaning the electric field is perpendicular to the surface, and $\vec{E} \circ \hat{n} = |\vec{E}| \cos(0°) = |\vec{E}|$. In this case, the component of $\vec{E}$ perpendicular to the surface is the entire length of $\vec{E}$.

The importance of the electric field component normal to the surface will become clear when you consider electric flux. To do that, you should make sure you understand the meaning of the surface integral in Gauss's law.

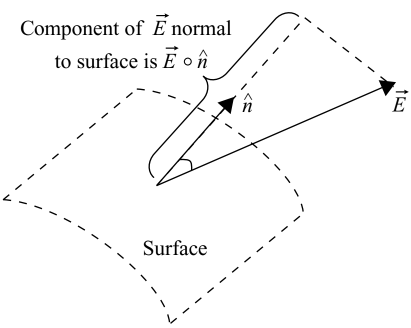


Figure 1.4 Projection of $\vec{E}$ onto direction of $\hat{n}$.

$\int_S ()da$ The surface integral

Many equations in physics and engineering – Gauss's law among them – involve the area integral of a scalar function or vector field over a specified surface (this type of integral is also called the "surface integral"). The time you spend understanding this important mathematical operation will be repaid many times over when you work problems in mechanics, fluid dynamics, and electricity and magnetism (E&M).

The meaning of the surface integral can be understood by considering a thin surface such as that shown in Figure 1.5. Imagine that the area density (the mass per unit area) of this surface varies with *x* and *y*, and you want to determine the total mass of the surface. You can do this by dividing the surface into two-dimensional segments over each of which the area density is approximately constant.

For individual segments with area density σ_i and area dA_i, the mass of each segment is $\sigma_i\, dA_i$, and the mass of the entire surface of N segments is given by $\sum_{i=1}^{N} \sigma_i\, dA_i$. As you can imagine, the smaller you make the area segments, the closer this gets to the true mass, since your approximation of constant σ is more accurate for smaller segments. If you let the segment area dA approach zero and N approach infinity, the summation becomes integration, and you have

$$\text{Mass} = \int_S \sigma(x, y)\, dA.$$

This is the area integral of the scalar function $\sigma(x, y)$ over the surface S. It is simply a way of adding up the contributions of little pieces of a function (the density in this case) to find a total quantity. To understand the integral form of Gauss's law, it is necessary to extend the concept of the surface integral to vector fields, and that's the subject of the next section.

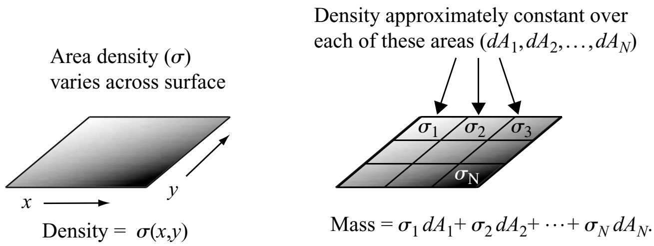


Figure 1.5 Finding the mass of a variable-density surface.

$\int_S \vec{A} \circ \hat{n}\, da$ The flux of a vector field

In Gauss's law, the surface integral is applied not to a scalar function (such as the density of a surface) but to a vector field. What's a vector field? As the name suggests, a vector field is a distribution of quantities in space – a field – and these quantities have both magnitude and direction, meaning that they are vectors. So whereas the distribution of temperature in a room is an example of a scalar field, the speed and direction of the flow of a fluid at each point in a stream is an example of a vector field.

The analogy of fluid flow is very helpful in understanding the meaning of the "flux" of a vector field, even when the vector field is static and nothing is actually flowing. You can think of the flux of a vector field over a surface as the "amount" of that field that "flows" through that surface, as illustrated in Figure 1.6.

In the simplest case of a uniform vector field $\vec{A}$ and a surface S perpendicular to the direction of the field, the flux Φ is defined as the product of the field magnitude and the area of the surface:

$$\Phi = |\vec{A}| \times \text{surface area}. \tag{1.5}$$

This case is shown in Figure 1.6(a). Note that if $\vec{A}$ is perpendicular to the surface, it is parallel to the unit normal $\hat{n}$.

If the vector field is uniform but is not perpendicular to the surface, as in Figure 1.6(b), the flux may be determined simply by finding the component of $\vec{A}$ perpendicular to the surface and then multiplying that value by the surface area:

$$\Phi = \vec{A} \circ \hat{n} \times (\text{surface area}). \tag{1.6}$$

While uniform fields and flat surfaces are helpful in understanding the concept of flux, many E&M problems involve nonuniform fields and curved surfaces. To work those kinds of problems, you'll need to understand how to extend the concept of the surface integral to vector fields.

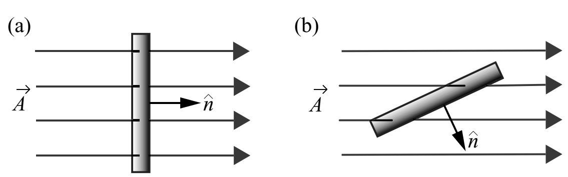


Figure 1.6 Flux of a vector field through a surface.