# Introduction

"There are 26 sheep and 10 goats on a ship. How old is the ship's captain?" This and similar tasks were given during the math lessons to primary school students in a number of European countries. More than 60% of students attempted to solve the problem by combining the given numbers, for example, by adding the number of sheep and the number of goats (Verschaffel, 1999). In our opinion, students' handling of the "Captain" problem is emblematic of the difficulties experienced by many students in the math classrooms because it clearly demonstrates that the students' main difficulty was not with mathematical knowledge but with more general cognitive functions that form prerequisites of mathematical reasoning. Students who blindly started to apply mathematical operations to the numbers given in the task ignored a host of cognitive operations that are needed for any sensible problem solving. They neither oriented themselves in the given data, nor compared or classified it. They also did not formulate the problem presented in this task, most probably because no one taught them the difference between the question ("How old . . . ") and the task's real problem. They apparently were not used to thinking of the tasks as having one solution or several or an unlimited number of correct solutions or no solution at all. For them, mathematics apparently appeared as an associative game where the winner correctly guesses which standard operation fits which one of the standard tasks.

In this book we attempt to demonstrate how rigorous mathematical thinking can be fostered through the development of cognitive tools and operations. Though our approach can be applied in any classroom, it seems to be particularly effective with socially disadvantaged and culturally different students. We will start with more general cognitive tools that are essential for all types of problem solving and then move to mathematically specific cognitive tools. Such an approach is based on our belief that although mathematics, as we know it today, represents an integration of elements from a number of cultures, it has its own unique culture that is distinctively different from "everyday ways" of doing things in various societies and cultures. Cognitive functions that appear naturally following the maturational path in one culture immediately reveal their culturally constructed nature once observed in children belonging to a different cultural group. Thus one cannot take for granted a certain type of cognitive development in students of a multicultural classroom. Their cognitive functions, both of a general and a more specifically mathematical nature, should be actively constructed during the educational process. Our rigorous mathematical thinking (RMT) model is based on two major theoretical approaches allowing such an active construction – Vygotsky's (1979; see also Kozulin, 1998a) theory of psychological tools and Feuerstein's (1990) theory of mediated learning experience.

Chapter 1 starts with a description of mathematical culture as having slowly developed over centuries from sociocultural needs systems rather than isolated, spontaneous efforts of individual scientists. A needs system is a set of internalized habits (autonomous ways of doing things), orientations (preferences and perspectives), and predispositions (inclinations and tendencies) that work together to provide the "blueprints" for actions and the meanings for developing know-how. Sociocultural needs systems are integrally and functionally bound to the life and "ways of living" of the human society. Their nature is an intertwining of affective and cognitive dimensions. Among the most prominent of these systems relevant to the mathematics culture are the needs for spatial and temporal orientation, determination of part/whole relationships, evaluation and establishment of constancy and change, order and organization, and so on. We then proceed to define the concepts of mathematical activity and mathematical knowledge.

The goal of mathematical learning is the appropriation of methods, tools, and conceptual principles of mathematical knowledge with efficient cognitive processing constituting an essential prerequisite of mathematical learning. Such a definition is based on the extension of Vygotsky's notion of learning activity (discussed in Chapter 3) to the domain of mathematical classroom learning. To achieve this objective we begin with identifying and elaborating specific criteria for determining which actions in the mathematics classroom meet the RMT standard. All of the following three criteria must be met for any action to qualify as a mathematical learning activity: (a) the action must contribute to creating a structural change in the students' understanding of mathematical knowledge; (b) the action must aim toward, and therefore be a part of, a systemic process for constructing a mathematics concept, because all concepts in mathematics are characterized as "scientific" according to Vygotsky (1986); and (c) the action must introduce the students to the language and rules of mathematics culture with regard to how things are done in mathematics.

Mathematical knowledge consists of organized, abstract systems of logical and precise understandings about patterns and relationships. These patterns and relationships may not originate in the everyday experience of the child, which, however, does not disqualify them as one of the sources for comprehending this experience mathematically. Mathematical knowledge exists at three levels: mathematical procedures and operations, mathematical concepts, and mathematical insights. Mathematical operations involve basic processes of organizing and manipulating mathematical information in meaningful ways that support and build important ideas and concepts. All mathematical concepts are "scientific" according to Vygotsky's (1986) definition of this term, that is, they are theoretical, systemic, and generative. Mathematical insight is derived from one or more of these conceptual understandings, forming relationships between these understandings, and constructing new ideas and/or applications.

In the RMT paradigm specific, well-defined cognitive processes drive mathematical operations and procedures. Mathematically specific cognitive tools, through their structure/function relationships, organize and integrate the use of cognitive processes and mathematical operations to systemically construct mathematical conceptual understandings. This rigorous practice of conceptual formation develops the students' habits of mind and a propensity for mathematical theoretical thinking and metacognition. These qualities position the student to make higher level reflections about patterns and relationships and create mathematical insights.

The next concept to be introduced is that of psychological tools. Mathematically specific psychological tools extend Vygotsky's (1979) notion of general psychological tools. Symbolic devices and schemes that have been developed through sociocultural needs to facilitate mathematical activity that, when internalized, become students' inner mathematical psychological tools. The structuring of these tools has slowly evolved over periods of time through collective, generalized purposes of the transitioning needs of the transforming cultures. Among the most prominent mathematically specific psychological tools are place value systems, number line, table, *x-y* coordinate plane, equations, and the language of mathematics. The problem in current mathematics instruction is that these devices are perceived by students as pieces of information or content rather than as "tools" or "instruments" to be used to organize and construct mathematical knowledge and understanding. Both the creation of such tools and their utilization develop, solicit, and further elaborate higher order mental processing that characterizes the dynamics of mathematical thinking. In this regard, the language of mathematics serves both as a tool and a higher order mental function. In the RMT paradigm the instrumentality of the language of mathematics can be viewed from the perspective of how it organizes and transforms students' everyday language and spontaneous concepts into more unified, abstract, and symbolic expressions.

Any genuine mathematical reasoning is rigorous. We define mental rigor as that quality of thought that reveals itself when students' critical engagement with material is driven by a strong, persistent, and inflexible desire to know and deeply understand. When this rigor is achieved, the learner is capable of functioning both in the immediate proximity as well as at some distance from the direct experience of the world and has an insight into the learning process, which has been described as metacognitive. This quality of engagement compels intellectual diligence, critical inquiry, and intense searching for truth – addressing the deep need to know and understand. Rigor describes the quality of being relentless in the face of challenge and complexity and having the motivation and self-discipline to persevere through a goal-oriented struggle. Rigorous thinking requires an intensive and aggressive mental engagement that dynamically seeks to create and sustain a higher quality of thought. Rigorous thinking can thus be characterized by sharpness in focus and perception; clarity and completeness in definition; delineation of critical attributes, precision, accuracy; and the depth of comprehension and understanding.

Chapter 2 focuses on the relationship between the RMT paradigm and the goals and objectives of mathematics education. The overarching goal of education in the United States is to prepare students to function as productive citizens in a highly industrialized and technical society. Since the 1960s there have been numerous attempts to reform education so that it provides a greater focus on scientific and mathematical literacy. One of the most recent attempts in this direction has been the standards-based movement, which has developed specific requirements for each learning subject at each grade level. However, these standards were often formulated in terms of the product of education rather than its process. Benchmarks were established that served as both frameworks and guidelines for curricula and anticipated milestones for student achievement. In spite of all of the good intentions of the standards movement, the current approach to teaching science and mathematics concepts in U.S. classrooms involves the presenting and eliciting of ready-made definitions with accompanying activities that, at best, produce little understanding and superficial applications. The focus in the applications usually does not extend beyond the mechanics or algorithms required for producing concrete answers. Students are not rigorously engaged in developing and manipulating the deeper structures of their thinking, nor are they challenged to synthesize from their own experiences and knowledge base the understanding necessary to induce the abstractions and generalizations that underlie science and mathematics concepts. Thus, many students complete courses in science and mathematics with the illusion of competency based on memory regurgitation. They do not build the understanding nor the flexible structures required for genuine transfer of learning and the creation of new knowledge in various contexts and situations. These surface experiences are not meaningful to students, do not promote science and mathematics competencies, and to some extent contribute to higher dropout rates.

To better understand stronger and weaker aspects of the standards movement, it is instructive to look at the difference between the American system and other systems of education. The study of Stigler and Hiebert (1997) demonstrated that U.S. 8th-grade students scored below their peers from 27 nations in mathematics and below their peers from 16 nations in science. The average international level, however, is also far from adequate. These and other research findings point to two gaps in students' mathematics and science academic achievement: overall, U.S. students perform below students from some other nations and students internationally perform well below expectations, particularly with regard to conceptual mastery. A third gap is the performance of minority students versus that of white students in the Untied States. The African American/white and Latino/white academic achievement gaps in mathematics in the United States widened in the 1990s after African American and Latino students' performance improved dramatically during the 1970s and 1980s.

For the standards movement to succeed, three critical needs should be addressed. First and foremost, U.S. students, and indeed all students, must develop the capability and drive to do rigorous higher order mathematical and scientific thinking. Second, high school students must develop a deep understanding of big ideas in mathematics and science and be able to apply them across various disciplines and in everyday living. Third, students must be able to communicate and express their mathematical and scientific thinking orally and in writing with precision and accuracy. It is imperative that the U.S. mathematics and science education enterprise make serious, substantial, and sustained investments in addressing these needs for real academic achievements and transfer of learning to take place for all students.

Chapter 3 demonstrates the relevance of Vygotsky's sociocultural theory for mathematics learning. For a long time, the predominant model of school learning was that of direct acquisition. Children were perceived as "containers" that must be filled with knowledge and skills. In time it became clear that the acquisition model is insufficient both theoretically and empirically. On the one hand, children have proved to be much more than passive recipients of information; on the other hand, students' independent acquisition has often led to the entrenchment of immature concepts and "misconceptions" as well as a neglect of important academic skills. A search for an alternative learning model brought to the fore such concepts as mediation, scaffolding, apprenticeship, and design of learning activities.

Vygotsky's (1986) theory stipulates that the development of the child's higher mental processes depends on the presence of mediating agents in the child's interaction with the environment. Vygotsky himself primarily emphasized symbolic tools-mediators appropriated by children in the context of particular sociocultural activities, the most important of which he considered to be formal education. Russian students of Vygotsky researched two additional types of mediation – mediation through another human being and mediation in a form of organized learning activity. Thus the acquisition model became transformed into a mediation model. Some mediational concepts such as scaffolding or apprenticeship appeared as a result of direct assimilation of Vygotsky's ideas; others like Feuerstein's (1990) mediated learning experience have been developed independently and only later became coordinated with the sociocultural theory.

In Vygotsky's sociocultural theory, cognitive development and learning are operationalized through the notion of psychological tools. Cultural-historical development of humankind created a wide range of higher order symbolic tools, including different signs, symbols, writing, formulae, and graphic organizers. Individual cognitive development and the progress in learning depend, according to Vygotsky, on the students' mastery of symbolic mediators and their appropriation and internalization in the form of inner psychological tools.

Mathematical education finds itself in a more difficult position vis-à-vis symbolic tools than other disciplines. On the one hand, the language of mathematical expressions and operations offers probably the greatest collection of potential psychological tools. On the other hand, because in mathematics everything is based on special symbolic language it is difficult for a student, and often also for a teacher, to distinguish between mathematical content and mathematical tools. One may classify psychological tools into two large groups. The first is general psychological tools that are used in a wide range of situations and in different disciplinary areas. Different forms of coding, lists, tables, plans, and pictures are examples of such general tools. One of the problems with the acquisition of these tools is that the educational system assumes that they are naturally and spontaneously acquired by children in their everyday life. As a result, general symbolic tools, such as tables or diagrams, appear in the context of teaching a particular curricular material and teachers rarely distinguish between difficulties caused by the students' lack of content knowledge and difficulties that originate in the students' poor mastery of symbolic tools themselves. The lack of symbolic tools becomes apparent only in special cases, such as a case of those immigrant students who come to the middle school without prior educational experience. For these students, a table is in no way a natural tool of their thought, because nothing in their previous experience is associated with this artifact.

Another of Vygotsky's concepts relevant to the task of developing RMT is the zone of proximal development (ZPD) – one of the most popular and, at the same time, most poorly understood of Vygotsky's theoretical constructs (see Chaiklin, 2003). From the perspective of math education, the developmental interpretation of ZPD calls for the analysis of those emerging psychological functions that provide the prerequisites of rigorous mathematical reasoning. Several questions can be asked here. For example, the emergence of which psychological functions is essential for successful mathematical reasoning at the child's next developmental period? What type of joint activity is most efficient in revealing and developing these functions in the child's ZPD? What characterizes the students' mathematically relevant ZPD at the primary, middle, and high school periods? These questions are directly related to the issue of the relationship between so-called cognitive education and mathematical education. There are reasons to believe that the students' mathematical failure is often triggered not by the lack of specific mathematical knowledge but by the absence of prerequisite cognitive functions of analysis, planning, and reflection. Cognitive intervention aimed at these emerging functions might be more effective in the long run than a simple drill of math operations that lack the underlying cognitive basis.

Implementation of Vygotskian sociocultural theory in the classroom is based on the concept of learning activity. Sociocultural theory makes an important distinction between generic learning and specially designed learning activity (LA). Formal learning becomes a dominant form of child's activity only at the primary school age and only in those societies that promote it. Generic learning, however, appears at all the developmental ages in the context of play, practical activity, apprenticeship, interpersonal interactions, and so on. In a somewhat tautological way, specially designed LA can be defined as a form of education that turns a child into a self-sufficient and self-regulated learner. In the LA classroom, learning ceases to be a mere acquisition of information and rules and becomes learning how to learn. Graduates of the LA classroom are capable of approaching any material as a problem and are ready to actively seek means for solving this problem. Three elements constitute the core of LA: analysis of the task, planning of action, and reflection. Although analysis and planning feature prominently in many educational models, reflection as a central element of the primary school education may justifiably be considered a "trademark" of the LA approach. According to Russian Vygotskians (Zuckerman, 2004), there are three major aspects of reflection to be developed in the primary school: (1) ability to identify goals of one's own and other people's actions, as well as methods and means for achieving these goals; (2) understanding other people's point of view by looking at the objects, processes, and problems from the perspective other than one's own; and (3) ability to evaluate oneself and identify strong points and shortcomings of one's own performance. For each one of the aspects of reflection, special forms of learning activity were developed.

Chapter 4 shows how the development of rigorous mathematical thinking benefits from the use of the concepts of mediated learning and cognitive functions developed by Feuerstein et al. (1980). Feuerstein et al. postulated that mediated learning experience (MLE) reflects a quality of interaction among the learner, the material, and the human mediator. The quality of this interaction can be achieved only if a number of MLE criteria are met. Among the most important of these criteria are intentionality and reciprocity of interaction, its transcendent character (i.e., having significance beyond a here-and-now situation), and the mediation of meaning. Studies that follow this paradigm focused predominantly on the impact of MLE on the child's formation of cognitive prerequisites of learning and on the consequences of the absence or insufficient amount of MLE for the child's cognitive development.

The RMT theory purports that cognitive processes are formed through the appropriation, internalization, and utilization of psychological tools through the application of the MLE interactional dynamic. It is here that the RMT theory is informed by the unique synthesis of constructs from Feuerstein's theory of MLE and Vygotsky's sociocultural theory, particularly with regard to his emphasis that cultural symbolic artifacts become mediators of higher order cognitive processes. Vygotsky insisted that this process takes place through transformation of natural psychological functions into higher level culturally oriented psychological functions. For this process to be effective, the appropriation and internalization of these symbolic devices should be accomplished through the application of the three central or universal criteria of MLE – (1) intentionality/reciprocity, (2) transcendence, and (3) mediation of meaning.

One of the primary roles of MLE is to guide and nurture students to construct and internalize cognitive functions forming prerequisites of efficient learning activity. In the RMT paradigm these cognitive functions provide the foundation for and generate the mechanisms of rigorous thinking that become catalysts and building blocks for concept formation. We believe that students develop these cognitive functions through the appropriation, internalization, and use of psychological tools. A cognitive function is a specific and deliberate thinking action that the student executes with awareness and intention. There are two broad aspects of a cognitive function – the conceptual component and the action component – that work in relationship to each other to provide the cognitive function with its integrity as a distinct mental activity or psychological process. Embedded in this description is the notion that every cognitive function has a structure/purpose or structure/function relationship.

The conceptual component provides a “steering” mechanism to the mental activity by defining or giving description to the nature of the action that is taking place when the function is executed. For example, the cognitive function of comparing conceptually involves similarities and differences between two or more items. The action component of comparing is the mental action of looking for or searching for the attributes that the items share or have in common and those attributes that they do not have in common. In other words, comparing is the mental act of carrying out a search between or among two or more items that is guided by an identification of similar and different attributes the items possess.

These two broad components of a cognitive function give it specificity or distinction while lending it the capacity to intimately network, operationally, with other functions. For example, while comparing demands the forming of relationships and vice versa, the two cognitive functions are distinct and different. It is this contradistinction in nature that provides the foundation to the mechanism underlying concept formation through cognitive processing, supporting the notion that cognitive functions are tools of conceptual development.

The Feuerstein et al. (1980) *instrumental enrichment* (IE) cognitive intervention program offers one of the richest sources for the acquisition of symbolic tools and operations associated with them. The program demonstrated its effectiveness in significantly improving problem-solving skills in learning disabled, underachieving, and culturally different students (see Kozulin, 2000). The IE program includes 14 booklets of paper-and-pencil tasks that cover such areas as analytic perception, comparisons, categorization, orientation in space and time, and syllogisms. These booklets are called “instruments” because they help to “repair” a number of deficient cognitive functions.

Essential cognitive functions or specific thinking actions needed to construct any standards-based mathematical concept can be systemically developed through the IE program. This systemic development is promoted by three factors. First, the content of each of the 14 instruments is designed to support the construction of each of these cognitive functions. Although the instruments and their pages are different with regard to appearance of stimuli and/or levels of complexity or abstraction, each page practically provides the opportunity to deepen the construction of each cognitive function. For example, essential cognitive functions to start building and deepening conceptual understanding of variable and functional relationships between variables are conserving constancy, comparing, analyzing, forming relationships, and labeling. Each of these cognitive functions must be mediated to students to perform the tasks in instruments or units of tasks such as "Organization of Dots," "Orientation in Space," "Analytic Perception," and "Numerical Progressions."

A second factor is that the organization of the IE material and the activities are designed in such a way that any single task in one unit is related to the whole system of tasks in that unit. For example, all tasks of the "Organization of Dots" unit is of the same nature – an unorganized cluster of dots must be investigated to determine how to organize them by projecting virtual relationships. Each task in this unit requires analyzing a set of models that must be appropriated as psychological tools to compare and form relationships to carry out these projections. Each set of models is different on each page and progresses in complexity from the first page to the last page. When students practice use of the cognitive functions through these progressive levels of rigor the robustness of the cognitive functions is systemically developed.

A third factor that leads to the systemic development of cognitive functions through the IE program is that mediating students through the structure of a unit of tasks demands an organized approach that leads to the discovery of general cognitive principles and strategies. This element contributes to the development of theoretical thinking in students.

One of the better documented successes of the IE program is its ability to help culturally different students to acquire symbolic tools and learning strategies that were absent in their native culture but are essential in the modern technological society. From the foundational studies of Feuerstein et al. (1980) with immigrant students from North Africa to more recent research with immigrant students from Ethiopia (Kozulin, 2005a) it has been demonstrated that students' psychological functions are highly modifiable and can be radically transformed through the application of the IE program.

Chapter 5 demonstrates the "mechanics" of creating rigorous mathematical thinking through combination of Feuerstein's IE with fostering in students the development of mathematically specific psychological tools. Although mathematics is indeed the study of patterns and relationships, the need for