Dynamics, Statistics and Projective Geometry of Galois Fields

V. I. Arnold reveals some unexpected connections between such apparently unrelated theories as Galois fields, dynamical systems, ergodic theory, statistics, chaos and the geometry of projective structures on finite sets. The author blends experimental results with examples and geometrical explorations to make these findings accessible to a broad range of mathematicians, from undergraduate students to experienced researchers.

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Some words about Vladimir Igorevich Arnold

As scientist, Vladimir Igorevich Arnold was among the most influential and greatest mathematicians of the XX century. His discoveries, conjectures and challenging problems strongly influenced the development and determined the modern state of such domains in mathematics as singularity theory, dynamical systems, real algebraic geometry, symplectic and contact geometry, symplectic and contact topology, KAM theory, qualitative theory of differential equations, mechanics and many others. According to the data of 2009, he was the most cited Russian scientist, being cited in papers of mathematics, physics, astronomy, chemistry, biology and even medicine. A small planet discovered in 1981, registered under # 10031, was named Vladarnolda after him – making him the only mathematician who has had such a distinction in life! Lots of mathematicians around the world are and have been publishing scientific papers on his conjectures and problems.

The authors of these lines were his students and are representatives of Arnold's Moscow and Paris schools, cities where he was professor and had a very active seminar, with lots of students. As professor, Arnold was doing extremely careful reading and critical corrections of our texts, pointing out all words or sentences that could be misunderstood. Sometimes, he was proposing deeper statements of the results, conjecturing new "theorems" (whose statements were often true, with slight modifications,) or improving the general redaction of the text. In particular, the first article of each one of us was almost entirely written by him (as his first paper was almost entirely written by Kolmogorov): he corrected the article three or four times, the size of his corrections being each time similar to the size of the article! In his seminar he was always proposing new problems to us, often leading to new topics. To prepare his students (or any mathematician attending to his seminar) to work on his problems, sometimes he explained the interconnections of those problems with different domains of mathematics and sketched the ways in which such

connections could be used. Other times he explained the essential elements of the corresponding theory, pointing out the possible diculties and making a proper choice and detailed study of relevant examples (he claimed that examples teach more than a formal proof of a result). After Arnold's problems a great number of publications were written by his students or by other mathematicians participating in his seminar. For several of those articles the main idea was due to him, but he never signed any such paper, nor any paper of his students. Besides his brilliant and visionary mind, Arnold was extremely generous. We learnt from him much more than mathematics and to be his student was a fascinating, enriching and life-changing experience.

Arnold's qualities as scientist and professor are reflected in his numerous books, many of them forming a golden fund of mathematical educational literature. His special style of writing (very easy to recognise and dicult to reproduce) is an amazing unity of clearness and profoundness that allowed to him to explain in an accessible way the theories standing on the very foreground of modern science, and in which a committed avoidance of useless and redundant formalism is a point of principle. Following the arguments of Arnold any thoughtful reader can readily reconstruct the details corresponding to his or her mathematical background and conception. Reading of his books turns into a fascinating pastime that is almost impossible to interrupt.

Arnold's books are equally interesting and useful to both working mathematicians and physicists as well as to students and teachers. Such books as "Ordinary Differential Equations" or "Mathematical Methods of Classical Mechanics" became bestsellers of mathematical literature. Arnold passed away the 3rd of July 2010 when the present book was already prepared for publication. We hope that it will also find a delightful response of many readers and will stimulate them to read Arnold's mathematical literature.

Maxim Kazarian and Ricardo Uribe-Vargas

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Preface

This book derives from a 2-hour-long presentation to Moscow high-school students at the Moscow State (Lomonosov) University MGU, in November 2004. It is a translation from the Russian of *The Dynamics, Statistics and Projective Geometry of Galois Fields*[†], which was itself based on the earlier article *Geometry and Dynamics of Galois Fields*.[‡] It describes some astonishing recent discoveries of the relations between Galois fields, dynamical systems, ergodic theory, statistics and chaos, as well as of the geometry of projective structures on finite sets.

Most of these recent discoveries encapsulated empirical studies, and some of the conjectures suggested by these numerical experiments are still unproved, despite the fact that their simple statements make them quite accessible to high-school students (who can study them empirically, thanks to computers).

Together with these continuing empirical studies, it would be nice to investigate some of the remaining theoretical questions, such as the natural problem of the intrinsic characterisation of projective permutations among all the permutations of a finite set. We ought to be able to understand those geometrical features of some special permutations of a dozen points that make these special permutations projective, thereby distinguishing them from non-projective permutations.

The author thanks the audience for many helpful remarks and hopes to extend the collaboration with the readers of the present book. The author looks forward to there being many contributions to this young domain of mathematics

[†] Moscow Center for Continuous Mathematical Education, Moscow, 72pp.
[‡] *Russian Mathematical Surveys*, **59** (6), (2004) (pages 23–40 in the Russian version).

(including, one hopes, the discovery of applications of Galois fields beyond mathematics).

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