A First Course in Mathematical Analysis

Mathematical Analysis (often called Advanced Calculus) is generally found by students to be one of their hardest courses in Mathematics. This text uses the so-called sequential approach to continuity, differentiability and integration to make it easier to understand the subject.

Topics that are generally glossed over in the standard Calculus courses are given careful study here. For example, what exactly is a ‘continuous’ function? And how exactly can one give a careful definition of ‘integral’? This latter is often one of the mysterious points in a Calculus course – and it is quite tricky to give a rigorous treatment of integration!

The text has a large number of diagrams and helpful margin notes, and uses many graded examples and exercises, often with complete solutions, to guide students through the tricky points. It is suitable for self study or use in parallel with a standard university course on the subject.

Cambridge University Press
978-0-521-68424-8 - A First Course in Mathematical Analysis
David Alexander Brannan
Table of Contents
More information

Cambridge University Press
978-0-521-68424-8 - A First Course in Mathematical Analysis
David Alexander Brannan
Table of Contents
More information

A First Course in Mathematical Analysis

DAVID ALEXANDER BRANNAN

Published in association with The Open University

The Open University

CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9780521864398

First published 2006
6th printing 2015

Printed in the United Kingdom by Clays, St Ives plc.

A catalogue record for this publication is available from the British Library

ISBN-13 978-0-521-86439-8 hardback
ISBN-13 978-0-521-68424-8 paperback

To my wife *Margaret*
and my sons *David*, *Joseph* and *Michael*

Cambridge University Press
978-0-521-68424-8 - A First Course in Mathematical Analysis
David Alexander Brannan
Table of Contents
More information

Cambridge University Press
978-0-521-68424-8 - A First Course in Mathematical Analysis
David Alexander Brannan
Table of Contents
More information

Contents

Cambridge University Press
978-0-521-68424-8 - A First Course in Mathematical Analysis
David Alexander Brannan
Table of Contents
More information

Preface

Analysis is a central topic in Mathematics, many of whose branches use key analytic tools. Analysis also has important applications in Applied Mathematics, Physics and Engineering, where a good appreciation of the underlying ideas of Analysis is necessary for a modern graduate.

Changes in the school curriculum over the last few decades have resulted in many students finding Analysis very difficult. The author believes that Analysis nowadays has an unjustified reputation for being hard, caused by the traditional university approach of providing students with a highly polished exposition in lectures and associated textbooks that make it impossible for the average learner to grasp the core ideas. Many students end up agreeing with the German poet and philosopher Goethe who wrote that 'Mathematicians are like Frenchmen: whatever you say to them, they translate into their own language, and forthwith it is something entirely different!'

Johann Wolfgang von Goethe (1749–1832) is said to have studied all areas of science of his day except mathematics – for which he had no aptitude.

Since 1971, the Open University in United Kingdom has taught Mathematics to students in their own homes via specially written correspondence texts, and has traditionally given Analysis a central position in its curriculum. Its philosophy is to provide clear and complete explanations of topics, and to teach these in a way that students can understand without much external help. As a result, students should be able to learn, and to enjoy learning, the key concepts of the subject in an uncluttered way. This book arises from correspondence texts for its course *Introduction to Pure Mathematics*, that has now been studied successfully by over ten thousand students.

This book is therefore different from most Mathematics textbooks! It adopts a student-friendly approach, being designed for study by a student on their own OR in parallel with a course that uses as set text either this text or another text. But this is the text that the student is likely to use to learn the subject from. The author hopes that readers will gain enormous pleasure from the subject's beauty and that this will encourage them to undertake further study of Mathematics!

Once a student has grasped the principal notions of *limit* and *continuous function* in terms of inequalities involving the three symbols ε, X and δ, they will quickly understand the unity of areas of Analysis such as limits, continuity, differentiability and integrability. Then they will thoroughly enjoy the beauty of some of the arguments used to prove key theorems – whether their proofs are short or long.

Calculus is the initial study of limits, continuity, differentiation and integration, where functions are assumed to be well-behaved. Thus all functions continuous on an interval are assumed to be differentiable at most points in the interval, and so on. However, Mathematics is not that simple! For example, there exist functions that are continuous everywhere on $\mathbb{R}$, but differentiable nowhere on $\mathbb{R}$; this discovery by Karl Weierstrass in 1872 caused a sensation in the mathematical community. In *Analysis* (sometimes called *Advanced Calculus*) we make no assumptions about the behaviour of functions – and the result is that we sometimes come across real surprises!

The book has two principal features in its approach that make it stand out from among other Analysis texts.

Firstly, this book uses the 'sequential approach' to Analysis. All too often students starting on the subject find that they cannot grasp the significance of both ε and δ simultaneously. This means that the whole underlying idea about what is happening is lost, and the student takes a very long time to master the topic – or, in many cases in fact, never masters the topic and acquires a strong dislike of it. In the sequential approach they proceed at a more leisurely pace to understand the notion of limit using ε and X – to handle convergent sequences – before coming across the other symbol δ, used in conjunction with ε to handle continuous functions. This approach avoids the conventional student horror at the perceived 'difficulty of Analysis'. Also, it avoids the necessity to re-prove broadly similar results in a range of settings – for example, results on the sum of two sequences, of two series, of two continuous functions and of two differentiable functions.

Secondly, this book makes great efforts to teach the ε–δ approach too. After students have had a first pass at convergence of sequences and series and at continuity using 'the sequential approach', they then meet 'the ε–δ approach', explained carefully and motivated by a clear 'ε–δ game' discussion. This makes the new approach seem very natural, and this is motivated by using each approach in later work in the appropriate situation. By the end of the book, students should have a good facility at using both the sequential approach and the ε–δ approach to proofs in Analysis, and should be better prepared for later study of Analysis than students who have acquired only a weak understanding of the conventional approach.

Outline of the content of the book

In Chapter 1, we define *real numbers* to be decimals. Rather than give a heavy discussion of *least upper bound* and *greatest lower bound*, we give an introduction to these matters sufficient for our purposes, and the full discussion is postponed to Chapter 7, where it is more timely. We also study *inequalities*, and their properties and proofs.

In Chapter 2, we define *convergent sequences* and examine their properties, basing the discussion on the notion of *null sequences*, which simplifies matters considerably. We also look at *divergent sequences*, sequences defined by *recurrence formulas* and particular sequences which converge to π and e.

In Chapter 3, we define *convergent infinite series*, and establish a number of tests for determining whether a given series is convergent or divergent. We demonstrate the equivalence of the two definitions of the *exponential function* $x \mapsto e^x$, and prove that the number e is irrational.

We define e^x as $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ and as $\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

In Chapter 4, we define carefully what we mean by a *continuous function*, in terms of sequences, and establish the key properties of continuous functions. We also give a rigorous definition of the *exponential function* $x \mapsto a^x$.

In Chapter 5, we define the *limit of a function* as x tends to c or as x tends to ∞ in terms of the convergence of sequences. Then we introduce the ε–δ definitions of limit and continuity, and check that these are equivalent to the earlier definitions in terms of sequences. We also look briefly at *uniform continuity*.

In Chapter 6, we define what we mean by a *differentiable function*, using *difference quotients* $Q(h)$; this enables us to use our earlier results on limits to prove corresponding results for differentiable functions. We establish some interesting properties of differentiable functions. Finally, we construct the *Blancmange function* that is continuous everywhere on $\mathbb{R}$, but differentiable nowhere on $\mathbb{R}$.

In Chapter 7, we give a careful definition of what we mean by an *integrable function*, and establish a number of related criteria for establishing whether a given function is integrable or not. Our integral is the so-called *Riemann integral*, defined in terms of upper and lower Riemann sums. We check the standard properties of integrals and verify a number of standard approaches for calculating definite integrals. Then we give a number of applications of integrals to limits of certain sequences and series and prove *Stirling's Formula*.

Stirling's Formula says that, for large n, $n!$ is 'roughly' $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, in a sense that we explain.

Finally, in Chapter 8, we study the convergence and properties of *power series*. The chapter ends with a marvellous proof of the irrationality of the number π that uses a whole range of the techniques that have been met in the previous chapters.

For completeness and for students' convenience, we give a brief guide to our notation for sets and functions, together with a brief indication of the logic involved in proofs in Mathematics (in particular, the Principle of Mathematical Induction) in Appendix 1. Appendix 2 contains a list of standard derivatives and primitives, and Appendix 3 the first 1000 decimal places in the values of the numbers $\sqrt{2}$, e and π. Appendix 4 contains full solutions to all the problems set during each chapter.

Solutions are not given to the exercises at the end of each chapter, however. Lecturers/instructors may wish to use these exercises in homework assignments.

Study guide

This book assumes that students have a fair understanding of Calculus. The assumptions on technical background are deliberately kept slight, however, so that students can concentrate on the newer aspects of the subject 'Analysis'.

Most students will have met some of the material in the early chapters previously. Although this means that they can therefore proceed fairly quickly through some sections, it does NOT mean that those sections can be ignored – each section contains important ideas that are used later on and most include something new or have a different emphasis.

Most chapters are divided into five or six sections (each often further divided into sub-sections); sections are numbered using two digits (such as 'Section 3.2') and sub-sections using three digits (such as 'Sub-section 3.2.4'). Generally a section is considered to be about one evening's work for an average student.

Chapter 7 on Integration is arguably the highlight of the book. However, it contains some rather complicated mathematical arguments and proofs.

Therefore, when reading Chapter 7, it is important not to get bogged down in details, but to keep progressing through the key ideas, and to return later on to reading the things that were left out at the earlier reading. Most students will require three or four passes at this chapter before having a good idea of most of it.

We use wide pages with a large number of margin notes in which we place teaching comments and some diagrams to aid in the understanding of particular points in arguments. We also provide advice on which proofs to omit on a first study of the topic; it is important for the student NOT to get bogged down in a technical discussion or a proof until they have a good idea of the message contained in the result and the situations in which it can be used. Therefore clear encouragement is given on which portions of the text to leave till later, or to simply skim on a first reading.

It is important to read the margin notes!

This signposting benefits students greatly in the author's experience.

The end of the proof of a Theorem is indicated by a solid symbol '■' and the end of the solution of a worked Example by a hollow symbol '□'. There are many worked examples within the text to explain the concepts being taught, together with a good stock of problems to reinforce the teaching. The solutions are a key part of the teaching, and tackling them on your own and then reading our version of the solution is a key part of the learning process.

Tackling the problems is a good use of your time, not something to skimp.

No one can learn Mathematics by simply reading – it is a 'hands on' activity. The reader should not be afraid to draw pictures to illustrate what seems to be happening to a sequence or a function, to get a feeling for its behaviour. A wise old man once said that 'A picture is worth a thousand words!'. A good picture may even suggest a method of proof. However, at the same time it is important not to regard a picture on its own as a proof of anything; it may illustrate just one situation that can arise and miss many other possibilities!

It is important NOT to become discouraged if a topic seems difficult. It took mathematicians hundreds of years to develop Analysis to its current polished state, so it may take the reader a few hours at several sittings to really grasp the more complex or subtle ideas.

Acknowledgements

The material in the Open University course on which this book is based was contributed to in some way by many colleagues, including Phil Rippon, Robin Wilson, Andrew Brown, Hossein Zand, Joan Aldous, Ian Harrison, Alan Best, Alison Cadle and Roberta Cheriyan. Its eventual appearance in book form owes much to Lynne Barber.

Without the forebearance of my family, the writing of the book would have been impossible.