MATHEMATICAL METHODS FOR PHYSICS
AND ENGINEERING

The third edition of this highly acclaimed undergraduate textbook is suitable for teaching all the mathematics ever likely to be needed for an undergraduate course in any of the physical sciences. As well as lucid descriptions of all the topics covered and many worked examples, it contains more than 800 exercises. A number of additional topics have been included and the text has undergone significant reorganisation in some areas. New stand-alone chapters:

- give a systematic account of the ‘special functions’ of physical science
- cover an extended range of practical applications of complex variables including WKB methods and saddle-point integration techniques
- provide an introduction to quantum operators.

Further tabulations, of relevance in statistics and numerical integration, have been added. In this edition, all 400 odd-numbered exercises are provided with complete worked solutions in a separate manual, available to both students and their teachers; these are in addition to the hints and outline answers given in the main text. The even-numbered exercises have no hints, answers or worked solutions and can be used for unaided homework; full solutions to them are available to instructors on a password-protected website.

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MATHEMATICAL METHODS FOR PHYSICS AND ENGINEERING

THIRD EDITION

K. F. RILEY, M. P. HOBSON and S. J. BENCE
## Contents

Preface to the third edition xx
Preface to the second edition xxiii
Preface to the first edition xxv

1 Preliminary algebra 1
   1.1 Simple functions and equations 1
      Polynomial equations; factorisation; properties of roots
   1.2 Trigonometric identities 10
      Single angle; compound angles; double- and half-angle identities
   1.3 Coordinate geometry 15
   1.4 Partial fractions 18
      Complications and special cases
   1.5 Binomial expansion 25
   1.6 Properties of binomial coefficients 27
   1.7 Some particular methods of proof 30
      Proof by induction; proof by contradiction; necessary and sufficient conditions
   1.8 Exercises 36
   1.9 Hints and answers 39

2 Preliminary calculus 41
   2.1 Differentiation 41
      Differentiation from first principles; products; the chain rule; quotients;
      implicit differentiation; logarithmic differentiation; Leibnitz' theorem; special
      points of a function; curvature; theorems of differentiation

v
## CONTENTS

2.2 Integration 59

Integration from first principles; the inverse of differentiation; by inspection; sinusoidal functions; logarithmic integration; using partial fractions; substitution method; integration by parts; reduction formulae; infinite and improper integrals; plane polar coordinates; integral inequalities; applications of integration

2.3 Exercises 76

2.4 Hints and answers 81

3 Complex numbers and hyperbolic functions 83

3.1 The need for complex numbers 83

3.2 Manipulation of complex numbers 85

Addition and subtraction; modulus and argument; multiplication; complex conjugate; division

3.3 Polar representation of complex numbers 92

Multiplication and division in polar form

3.4 de Moivre’s theorem 95

Trigonometric identities; finding the nth roots of unity; solving polynomial equations

3.5 Complex logarithms and complex powers 99

3.6 Applications to differentiation and integration 101

3.7 Hyperbolic functions 102

Definitions; hyperbolic–trigonometric analogies; identities of hyperbolic functions; solving hyperbolic equations; inverses of hyperbolic functions; calculus of hyperbolic functions

3.8 Exercises 109

3.9 Hints and answers 113

4 Series and limits 115

4.1 Series 115

4.2 Summation of series 116

Arithmetic series; geometric series; arithmetico-geometric series; the difference method; series involving natural numbers; transformation of series

4.3 Convergence of infinite series 124

Absolute and conditional convergence; series containing only real positive terms; alternating series test

4.4 Operations with series 131

4.5 Power series 131

Convergence of power series; operations with power series

4.6 Taylor series 136

Taylor’s theorem; approximation errors; standard Maclaurin series

4.7 Evaluation of limits 141

4.8 Exercises 144

4.9 Hints and answers 149
## CONTENTS

### 5 Partial differentiation 151

5.1 Definition of the partial derivative 151
5.2 The total differential and total derivative 153
5.3 Exact and inexact differentials 155
5.4 Useful theorems of partial differentiation 157
5.5 The chain rule 157
5.6 Change of variables 158
5.7 Taylor’s theorem for many-variable functions 160
5.8 Stationary values of many-variable functions 162
5.9 Stationary values under constraints 167
5.10 Envelopes 173
5.11 Thermodynamic relations 176
5.12 Differentiation of integrals 178
5.13 Exercises 179
5.14 Hints and answers 185

### 6 Multiple integrals 187

6.1 Double integrals 187
6.2 Triple integrals 190
6.3 Applications of multiple integrals 191
   *Areas and volumes; masses, centres of mass and centroids; Pappus’ theorems; moments of inertia; mean values of functions*
6.4 Change of variables in multiple integrals 199
   *Change of variables in double integrals; evaluation of the integral \( I = \int_{-\infty}^{\infty} e^{-x^2} \, dx \); change of variables in triple integrals; general properties of Jacobians*
6.5 Exercises 207
6.6 Hints and answers 211

### 7 Vector algebra 212

7.1 Scalars and vectors 212
7.2 Addition and subtraction of vectors 213
7.3 Multiplication by a scalar 214
7.4 Basis vectors and components 217
7.5 Magnitude of a vector 218
7.6 Multiplication of vectors 219
   *Scalar product; vector product; scalar triple product; vector triple product*
CONTENTS

7.7 Equations of lines, planes and spheres 226
7.8 Using vectors to find distances 229
   Point to line; point to plane; line to line; line to plane
7.9 Reciprocal vectors 233
7.10 Exercises 234
7.11 Hints and answers 240

8 Matrices and vector spaces 241
8.1 Vector spaces 242
   Basis vectors; inner product; some useful inequalities
8.2 Linear operators 247
8.3 Matrices 249
8.4 Basic matrix algebra 250
   Matrix addition; multiplication by a scalar; matrix multiplication
8.5 Functions of matrices 255
8.6 The transpose of a matrix 255
8.7 The complex and Hermitian conjugates of a matrix 256
8.8 The trace of a matrix 258
8.9 The determinant of a matrix 259
   Properties of determinants
8.10 The inverse of a matrix 263
8.11 The rank of a matrix 267
8.12 Special types of square matrix 268
   Diagonal; triangular; symmetric and antisymmetric; orthogonal; Hermitian and
   anti-Hermitian; unitary; normal
8.13 Eigenvectors and eigenvalues 272
   Of a normal matrix; of Hermitian and anti-Hermitian matrices; of a unitary
   matrix; of a general square matrix
8.14 Determination of eigenvalues and eigenvectors 280
   Degenerate eigenvalues
8.15 Change of basis and similarity transformations 282
8.16 Diagonalisation of matrices 285
8.17 Quadratic and Hermitian forms 288
   Stationary properties of the eigenvectors; quadratic surfaces
8.18 Simultaneous linear equations 292
   Range; null space; N simultaneous linear equations in N unknowns; singular
   value decomposition
8.19 Exercises 307
8.20 Hints and answers 314

9 Normal modes 316
9.1 Typical oscillatory systems 317
9.2 Symmetry and normal modes 322
## CONTENTS

9.3 Rayleigh–Ritz method 327  
9.4 Exercises 329  
9.5 Hints and answers 332  

10 Vector calculus 334  
10.1 Differentiation of vectors 334  
   Composite vector expressions; differential of a vector  
10.2 Integration of vectors 339  
10.3 Space curves 340  
10.4 Vector functions of several arguments 344  
10.5 Surfaces 345  
10.6 Scalar and vector fields 347  
10.7 Vector operators 347  
   Gradient of a scalar field; divergence of a vector field; curl of a vector field  
10.8 Vector operator formulae 354  
   Vector operators acting on sums and products; combinations of grad, div and curl  
10.9 Cylindrical and spherical polar coordinates 357  
10.10 General curvilinear coordinates 364  
10.11 Exercises 369  
10.12 Hints and answers 375  

11 Line, surface and volume integrals 377  
11.1 Line integrals 377  
   Evaluating line integrals; physical examples; line integrals with respect to a scalar  
11.2 Connectivity of regions 383  
11.3 Green’s theorem in a plane 384  
11.4 Conservative fields and potentials 387  
11.5 Surface integrals 389  
   Evaluating surface integrals; vector areas of surfaces; physical examples  
11.6 Volume integrals 396  
   Volumes of three-dimensional regions  
11.7 Integral forms for grad, div and curl 398  
11.8 Divergence theorem and related theorems 401  
   Green’s theorems; other related integral theorems; physical applications  
11.9 Stokes’ theorem and related theorems 406  
   Related integral theorems; physical applications  
11.10 Exercises 409  
11.11 Hints and answers 414  

12 Fourier series 415  
12.1 The Dirichlet conditions 415  
12.2 The Fourier coefficients 417
12.3 Symmetry considerations ........................................... 419
12.4 Discontinuous functions ........................................... 420
12.5 Non-periodic functions ........................................... 422
12.6 Integration and differentiation ................................... 424
12.7 Complex Fourier series ........................................... 424
12.8 Parseval’s theorem .................................................. 426
12.9 Exercises ............................................................. 427
12.10 Hints and answers ................................................... 431

13 Integral transforms .................................................. 433
13.1 Fourier transforms .................................................. 433
   The uncertainty principle; Fraunhofer diffraction; the Dirac δ-function; relation of
   the δ-function to Fourier transforms; properties of Fourier transforms; odd and even
   functions; convolution and deconvolution; correlation functions and energy spectra;
   Parseval’s theorem; Fourier transforms in higher dimensions
13.2 Laplace transforms .................................................. 453
   Laplace transforms of derivatives and integrals; other properties of Laplace
   transforms
13.3 Concluding remarks .................................................. 459
13.4 Exercises ............................................................. 460
13.5 Hints and answers ................................................... 466

14 First-order ordinary differential equations ......................... 468
14.1 General form of solution ........................................... 469
14.2 First-degree first-order equations ................................ 470
   Separable-variable equations; exact equations; inexact equations, integrating factors;
   linear equations; homogeneous equations; isobaric equations; Bernoulli’s equation;
   miscellaneous equations
14.3 Higher-degree first-order equations ............................... 480
   Equations soluble for p; for x; for y; Clairaut’s equation
14.4 Exercises ............................................................. 484
14.5 Hints and answers ................................................... 488

15 Higher-order ordinary differential equations ....................... 490
15.1 Linear equations with constant coefficients ..................... 492
   Finding the complementary function y_c(x); finding the particular integral
   y_p(x); constructing the general solution y_c(x) + y_p(x); linear recurrence
   relations; Laplace transform method
15.2 Linear equations with variable coefficients ..................... 503
   The Legendre and Euler linear equations; exact equations; partially known
   complementary function; variation of parameters; Green’s functions; canonical
   form for second-order equations

x
CONTENTS

15.3 General ordinary differential equations 518
  Dependent variable absent; independent variable absent; non-linear exact
  equations; isobaric or homogeneous equations; equations homogeneous in x
  or y alone; equations having \( y = Ae^x \) as a solution

15.4 Exercises 523

15.5 Hints and answers 529

16 Series solutions of ordinary differential equations 531
16.1 Second-order linear ordinary differential equations 531
  Ordinary and singular points

16.2 Series solutions about an ordinary point 535

16.3 Series solutions about a regular singular point 538
  Distinct roots not differing by an integer; repeated root of the indicial
  equation; distinct roots differing by an integer

16.4 Obtaining a second solution 544
  The Wronskian method; the derivative method; series form of the second solution

16.5 Polynomial solutions 548

16.6 Exercises 550

16.7 Hints and answers 553

17 Eigenfunction methods for differential equations 554
17.1 Sets of functions 556
  Some useful inequalities

17.2 Adjoint, self-adjoint and Hermitian operators 559

17.3 Properties of Hermitian operators 561
  Reality of the eigenvalues; orthogonality of the eigenfunctions; construction
  of real eigenfunctions

17.4 Sturm–Liouville equations 564
  Valid boundary conditions; putting an equation into Sturm–Liouville form

17.5 Superposition of eigenfunctions: Green’s functions 569

17.6 A useful generalisation 572

17.7 Exercises 573

17.8 Hints and answers 576

18 Special functions 577
18.1 Legendre functions 577
  General solution for integer \( \ell \); properties of Legendre polynomials

18.2 Associated Legendre functions 587

18.3 Spherical harmonics 593

18.4 Chebyshev functions 595

18.5 Bessel functions 602
  General solution for non-integer \( \nu \); general solution for integer \( \nu \); properties
  of Bessel functions

18.6 Spherical Bessel functions 614
CONTENTS

18.7 Laguerre functions 616
18.8 Associated Laguerre functions 621
18.9 Hermite functions 624
18.10 Hypergeometric functions 628
18.11 Confluent hypergeometric functions 633
18.12 The gamma function and related functions 635
18.13 Exercises 640
18.14 Hints and answers 646

19 Quantum operators 648
19.1 Operator formalism 648
Commutators 656
19.2 Physical examples of operators
Uncertainty principle; angular momentum; creation and annihilation operators 656
19.3 Exercises 671
19.4 Hints and answers 674

20 Partial differential equations: general and particular solutions 675
20.1 Important partial differential equations 676
The wave equation; the diffusion equation; Laplace's equation; Poisson's equation; Schrödinger's equation
20.2 General form of solution 680
20.3 General and particular solutions 681
First-order equations; inhomogeneous equations and problems; second-order equations
20.4 The wave equation 693
20.5 The diffusion equation 695
20.6 Characteristics and the existence of solutions 699
First-order equations; second-order equations
20.7 Uniqueness of solutions 705
20.8 Exercises 707
20.9 Hints and answers 711

21 Partial differential equations: separation of variables and other methods 713
21.1 Separation of variables: the general method 713
21.2 Superposition of separated solutions 717
21.3 Separation of variables in polar coordinates 725
Laplace's equation in polar coordinates; spherical harmonics; other equations in polar coordinates; solution by expansion; separation of variables for inhomogeneous equations
21.4 Integral transform methods 747
## CONTENTS

21.5 Inhomogeneous problems – Green’s functions | 751

*Similarities to Green’s functions for ordinary differential equations; general boundary-value problems; Dirichlet problems; Neumann problems*

21.6 Exercises | 767

21.7 Hints and answers | 773

22 Calculus of variations | 775

22.1 The Euler–Lagrange equation | 776

22.2 Special cases | 777

*F does not contain y explicitly; F does not contain x explicitly*

22.3 Some extensions | 781

*Several dependent variables; several independent variables; higher-order derivatives; variable end-points*

22.4 Constrained variation | 785

22.5 Physical variational principles | 787

*Fermat’s principle in optics; Hamilton’s principle in mechanics*

22.6 General eigenvalue problems | 790

22.7 Estimation of eigenvalues and eigenfunctions | 792

22.8 Adjustment of parameters | 795

22.9 Exercises | 797

22.10 Hints and answers | 801

23 Integral equations | 803

23.1 Obtaining an integral equation from a differential equation | 803

23.2 Types of integral equation | 804

23.3 Operator notation and the existence of solutions | 805

23.4 Closed-form solutions | 806

*Separable kernels; integral transform methods; differentiation*

23.5 Neumann series | 813

23.6 Fredholm theory | 815

23.7 Schmidt–Hilbert theory | 816

23.8 Exercises | 819

23.9 Hints and answers | 823

24 Complex variables | 824

24.1 Functions of a complex variable | 825

24.2 The Cauchy–Riemann relations | 827

24.3 Power series in a complex variable | 830

24.4 Some elementary functions | 832

24.5 Multivalued functions and branch cuts | 835

24.6 Singularities and zeros of complex functions | 837

24.7 Conformal transformations | 839

24.8 Complex integrals | 845
CONTENTS

24.9 Cauchy’s theorem 849
24.10 Cauchy’s integral formula 851
24.11 Taylor and Laurent series 853
24.12 Residue theorem 858
24.13 Definite integrals using contour integration 861
24.14 Exercises 867
24.15 Hints and answers 870

25 Applications of complex variables 871
25.1 Complex potentials 871
25.2 Applications of conformal transformations 876
25.3 Location of zeros 879
25.4 Summation of series 882
25.5 Inverse Laplace transform 884
25.6 Stokes’ equation and Airy integrals 888
25.7 WKB methods 895
25.8 Approximations to integrals 905
Level lines and saddle points; steepest descents; stationary phase
25.9 Exercises 920
25.10 Hints and answers 925

26 Tensors 927
26.1 Some notation 928
26.2 Change of basis 929
26.3 Cartesian tensors 930
26.4 First- and zero-order Cartesian tensors 932
26.5 Second- and higher-order Cartesian tensors 935
26.6 The algebra of tensors 938
26.7 The quotient law 939
26.8 The tensors $\delta_{ij}$ and $\epsilon_{ijk}$ 941
26.9 Isotropic tensors 944
26.10 Improper rotations and pseudotensors 946
26.11 Dual tensors 949
26.12 Physical applications of tensors 950
26.13 Integral theorems for tensors 954
26.14 Non-Cartesian coordinates 955
26.15 The metric tensor 957
26.16 General coordinate transformations and tensors 960
26.17 Relative tensors 963
26.18 Derivatives of basis vectors and Christoffel symbols 965
26.19 Covariant differentiation 968
26.20 Vector operators in tensor form 971

xiv
CONTENTS

26.21 Absolute derivatives along curves 975
26.22 Geodesics 976
26.23 Exercises 977
26.24 Hints and answers 982

27 Numerical methods 984
27.1 Algebraic and transcendental equations 985
\textit{Rearrangement of the equation; linear interpolation; binary chopping; Newton–Raphson method}
27.2 Convergence of iteration schemes 992
27.3 Simultaneous linear equations 994
\textit{Gaussian elimination; Gauss–Seidel iteration; tridiagonal matrices}
27.4 Numerical integration 1000
\textit{Trapezium rule; Simpson's rule; Gaussian integration; Monte Carlo methods}
27.5 Finite differences 1019
27.6 Differential equations 1020
\textit{Difference equations; Taylor series solutions; prediction and correction; Runge–Kutta methods; isoclines}
27.7 Higher-order equations 1028
27.8 Partial differential equations 1030
27.9 Exercises 1033
27.10 Hints and answers 1039

28 Group theory 1041
28.1 Groups 1041
\textit{Definition of a group; examples of groups}
28.2 Finite groups 1049
28.3 Non-Abelian groups 1052
28.4 Permutation groups 1056
28.5 Mappings between groups 1059
28.6 Subgroups 1061
28.7 Subdividing a group 1063
\textit{Equivalence relations and classes; congruence and cosets; conjugates and classes}
28.8 Exercises 1070
28.9 Hints and answers 1074

29 Representation theory 1076
29.1 Dipole moments of molecules 1077
29.2 Choosing an appropriate formalism 1078
29.3 Equivalent representations 1084
29.4 Reducibility of a representation 1086
29.5 The orthogonality theorem for irreducible representations 1090
# CONTENTS

29.6 Characters 1092
   Orthogonality property of characters

29.7 Counting irreps using characters 1095
   Summation rules for irreps

29.8 Construction of a character table 1100
29.9 Group nomenclature 1102
29.10 Product representations 1103
29.11 Physical applications of group theory 1105
   Bonding in molecules; matrix elements in quantum mechanics; degeneracy of
   normal modes; breaking of degeneracies

29.12 Exercises 1113
29.13 Hints and answers 1117

30  Probability 1119
30.1 Venn diagrams 1119
30.2 Probability 1124
   Axioms and theorems; conditional probability; Bayes' theorem
30.3 Permutations and combinations 1133
30.4 Random variables and distributions 1139
   Discrete random variables; continuous random variables
30.5 Properties of distributions 1143
   Mean; mode and median; variance and standard deviation; moments; central
   moments
30.6 Functions of random variables 1150
30.7 Generating functions 1157
   Probability generating functions; moment generating functions; characteristic
   functions; cumulant generating functions
30.8 Important discrete distributions 1168
   Binomial; geometric; negative binomial; hypergeometric; Poisson
30.9 Important continuous distributions 1179
   Gaussian; log-normal; exponential; gamma; chi-squared; Cauchy; Breit–
   Wigner; uniform
30.10 The central limit theorem 1195
30.11 Joint distributions 1196
   Discrete bivariate; continuous bivariate; marginal and conditional distributions
30.12 Properties of joint distributions 1199
   Means; variances; covariance and correlation
30.13 Generating functions for joint distributions 1205
30.14 Transformation of variables in joint distributions 1206
30.15 Important joint distributions 1207
   Multinomial; multivariate Gaussian
30.16 Exercises 1211
30.17 Hints and answers 1219
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>Statistics</td>
</tr>
<tr>
<td>31.1</td>
<td>Experiments, samples and populations</td>
</tr>
<tr>
<td>31.2</td>
<td>Sample statistics</td>
</tr>
<tr>
<td>31.3</td>
<td>Estimators and sampling distributions</td>
</tr>
<tr>
<td>31.4</td>
<td>Some basic estimators</td>
</tr>
<tr>
<td>31.5</td>
<td>Maximum-likelihood method</td>
</tr>
<tr>
<td>31.6</td>
<td>The method of least squares</td>
</tr>
<tr>
<td>31.7</td>
<td>Hypothesis testing</td>
</tr>
<tr>
<td>31.8</td>
<td>Exercises</td>
</tr>
<tr>
<td>31.9</td>
<td>Hints and answers</td>
</tr>
</tbody>
</table>

CONTENTS

31 Statistics 1221
31.1 Experiments, samples and populations 1221
31.2 Sample statistics 1222
Averages; variance and standard deviation; moments; covariance and correlation
31.3 Estimators and sampling distributions 1229
Consistency, bias and efficiency; Fisher's inequality; standard errors; confidence limits
31.4 Some basic estimators 1243
Mean; variance; standard deviation; moments; covariance and correlation
31.5 Maximum-likelihood method 1255
ML estimator; transformation invariance and bias; efficiency; errors and confidence limits; Bayesian interpretation; large-N behaviour; extended ML method
31.6 The method of least squares 1271
Linear least squares; non-linear least squares
31.7 Hypothesis testing 1277
Simple and composite hypotheses; statistical tests; Neyman–Pearson; generalised likelihood-ratio; Student's t; Fisher's F; goodness of fit
31.8 Exercises 1298
31.9 Hints and answers 1303
Index 1305
I am the very Model for a Student Mathematical

I am the very model for a student mathematical;
I’ve information rational, and logical and practical.
I know the laws of algebra, and find them quite symmetrical,
And even know the meaning of ‘a variate antithetical’.

I’m extremely well acquainted, with all things mathematical.
I understand equations, both the simple and quadratical.
About binomial theorems I’m teeming with a lot o’news,
With many cheerful facts about the square of the hypotenuse.

I’m very good at integral and differential calculus,
And solving paradoxes that so often seem to rankle us.
In short in matters rational, and logical and practical,
I am the very model for a student mathematical.

I know the singularities of equations differential,
And some of these are regular, but the rest are quite essential.
I quote the results of giants; with Euler, Newton, Gauss, Laplace,
And can calculate an orbit, given a centre, force and mass.

I can reconstruct equations, both canonical and formal,
And write all kinds of matrices, orthogonal, real and normal.
I show how to tackle problems that one has never met before,
By analogy or example, or with some clever metaphor.

I seldom use equivalence to help decide upon a class,
But often find an integral, using a contour o’er a pass.
In short in matters rational, and logical and practical,
I am the very model for a student mathematical.

When you have learnt just what is meant by ‘Jacobian’ and ‘Abelian’;
When you at sight can estimate, for the modal, mean and median;
When describing normal subgroups is much more than recitation;
When you understand precisely what is ‘quantum excitation’;

When you know enough statistics that you can recognise RV;
When you have learnt all advances that have been made in SVD;
And when you can spot the transform that solves some tricky PDE,
You will feel no better student has ever sat for a degree.

Your accumulated knowledge, whilst extensive and exemplary,
Will have only been brought down to the beginning of last century,
But still in matters rational, and logical and practical,
You’ll be the very model of a student mathematical.

KFR, with apologies to W.S. Gilbert

xix
Preface to the third edition

As is natural, in the four years since the publication of the second edition of this book we have somewhat modified our views on what should be included and how it should be presented. In this new edition, although the range of topics covered has been extended, there has been no significant shift in the general level of difficulty or in the degree of mathematical sophistication required. Further, we have aimed to preserve the same style of presentation as seems to have been well received in the first two editions. However, a significant change has been made to the format of the chapters, specifically to the way that the exercises, together with their hints and answers, have been treated; the details of the change are explained below.

The two major chapters that are new in this third edition are those dealing with ‘special functions’ and the applications of complex variables. The former presents a systematic account of those functions that appear to have arisen in a more or less haphazard way as a result of studying particular physical situations, and are deemed ‘special’ for that reason. The treatment presented here shows that, in fact, they are nearly all particular cases of the hypergeometric or confluent hypergeometric functions, and are special only in the sense that the parameters of the relevant function take simple or related values.

The second new chapter describes how the properties of complex variables can be used to tackle problems arising from the description of physical situations or from other seemingly unrelated areas of mathematics. To topics treated in earlier editions, such as the solution of Laplace’s equation in two dimensions, the summation of series, the location of zeros of polynomials and the calculation of inverse Laplace transforms, has been added new material covering Airy integrals, saddle-point methods for contour integral evaluation, and the WKB approach to asymptotic forms.

Other new material includes a stand-alone chapter on the use of coordinate-free operators to establish valuable results in the field of quantum mechanics; amongst
the physical topics covered are angular momentum and uncertainty principles. There are also significant additions to the treatment of numerical integration. In particular, Gaussian quadrature based on Legendre, Laguerre, Hermite and Chebyshev polynomials is discussed, and appropriate tables of points and weights are provided.

We now turn to the most obvious change to the format of the book, namely the way that the exercises, hints and answers are treated. The second edition of Mathematical Methods for Physics and Engineering carried more than twice as many exercises, based on its various chapters, as did the first. In its preface we discussed the general question of how such exercises should be treated but, in the end, decided to provide hints and outline answers to all problems, as in the first edition. This decision was an uneasy one as, on the one hand, it did not allow the exercises to be set as totally unaided homework that could be used for assessment purposes but, on the other, it did not give a full explanation of how to tackle a problem when a student needed explicit guidance or a model answer.

In order to allow both of these educationally desirable goals to be achieved, we have, in this third edition, completely changed the way in which this matter is handled. A large number of exercises have been included in the penultimate subsections of the appropriate, sometimes reorganised, chapters. Hints and outline answers are given, as previously, in the final subsections, but only for the odd-numbered exercises. This leaves all even-numbered exercises free to be set as unaided homework, as described below.

For the four hundred plus odd-numbered exercises, complete solutions are available, to both students and their teachers, in the form of a separate manual, Student Solutions Manual for Mathematical Methods for Physics and Engineering (Cambridge: Cambridge University Press, 2006); the hints and outline answers given in this main text are brief summaries of the model answers given in the manual. There, each original exercise is reproduced and followed by a fully worked solution. For those original exercises that make internal reference to this text or to other (even-numbered) exercises not included in the solutions manual, the questions have been reworded, usually by including additional information, so that the questions can stand alone.

In many cases, the solution given in the manual is even fuller than one that might be expected of a good student that has understood the material. This is because we have aimed to make the solutions instructional as well as utilitarian. To this end, we have included comments that are intended to show how the plan for the solution is formulated and have given the justifications for particular intermediate steps (something not always done, even by the best of students). We have also tried to write each individual substituted formula in the form that best indicates how it was obtained, before simplifying it at the next or a subsequent stage. Where several lines of algebraic manipulation or calculus are needed to obtain a final result, they are normally included in full; this should enable the
student to determine whether an incorrect answer is due to a misunderstanding of principles or to a technical error.

The remaining four hundred or so even-numbered exercises have no hints or answers, outlined or detailed, available for general access. They can therefore be used by instructors as a basis for setting unaided homework. Full solutions to these exercises, in the same general format as those appearing in the manual (though they may contain references to the main text or to other exercises), are available without charge to accredited teachers as downloadable pdf files on the password-protected website http://www.cambridge.org/9780521679718. Teachers wishing to have access to the website should contact solutions@cambridge.org for registration details.

In all new publications, errors and typographical mistakes are virtually unavoidable, and we would be grateful to any reader who brings instances to our attention. Retrospectively, we would like to record our thanks to Reinhard Gerndt, Paul Renteln and Joe Tenn for making us aware of some errors in the second edition. Finally, we are extremely grateful to Dave Green for his considerable and continuing advice concerning \LaTeX{}.

Ken Riley, Michael Hobson,
Cambridge, 2006
Preface to the second edition

Since the publication of the first edition of this book, both through teaching the material it covers and as a result of receiving helpful comments from colleagues, we have become aware of the desirability of changes in a number of areas. The most important of these is that the mathematical preparation of current senior college and university entrants is now less thorough than it used to be. To match this, we decided to include a preliminary chapter covering areas such as polynomial equations, trigonometric identities, coordinate geometry, partial fractions, binomial expansions, necessary and sufficient condition and proof by induction and contradiction.

Whilst the general level of what is included in this second edition has not been raised, some areas have been expanded to take in topics we now feel were not adequately covered in the first. In particular, increased attention has been given to non-square sets of simultaneous linear equations and their associated matrices. We hope that this more extended treatment, together with the inclusion of singular value matrix decomposition, will make the material of more practical use to engineering students. In the same spirit, an elementary treatment of linear recurrence relations has been included. The topic of normal modes has been given a small chapter of its own, though the links to matrices on the one hand, and to representation theory on the other, have not been lost.

Elsewhere, the presentation of probability and statistics has been reorganised to give the two aspects more nearly equal weights. The early part of the probability chapter has been rewritten in order to present a more coherent development based on Boolean algebra, the fundamental axioms of probability theory and the properties of intersections and unions. Whilst this is somewhat more formal than previously, we think that it has not reduced the accessibility of these topics and hope that it has increased it. The scope of the chapter has been somewhat extended to include all physically important distributions and an introduction to cumulants.
Statistics now occupies a substantial chapter of its own, one that includes systematic discussions of estimators and their efficiency, sample distributions and $t$- and $F$-tests for comparing means and variances. Other new topics are applications of the chi-squared distribution, maximum-likelihood parameter estimation and least-squares fitting. In other chapters we have added material on the following topics: curvature, envelopes, curve-sketching, more refined numerical methods for differential equations and the elements of integration using Monte Carlo techniques.

Over the last four years we have received somewhat mixed feedback about the number of exercises at the ends of the various chapters. After consideration, we decided to increase the number substantially, partly to correspond to the additional topics covered in the text but mainly to give both students and their teachers a wider choice. There are now nearly 800 such exercises, many with several parts. An even more vexed question has been whether to provide hints and answers to all the exercises or just to ‘the odd-numbered’ ones, as is the normal practice for textbooks in the United States, thus making the remainder more suitable for setting as homework. In the end, we decided that hints and outline solutions should be provided for all the exercises, in order to facilitate independent study while leaving the details of the calculation as a task for the student.

In conclusion, we hope that this edition will be thought by its users to be ‘heading in the right direction’ and would like to place on record our thanks to all who have helped to bring about the changes and adjustments. Naturally, those colleagues who have noted errors or ambiguities in the first edition and brought them to our attention figure high on the list, as do the staff at The Cambridge University Press. In particular, we are grateful to Dave Green for continued \LaTeXe advice, Susan Parkinson for copy-editing the second edition with her usual keen eye for detail and flair for crafting coherent prose and Alison Woollatt for once again turning our basic \LaTeX into a beautifully typeset book. Our thanks go to all of them, though of course we accept full responsibility for any remaining errors or ambiguities, of which, as with any new publication, there are bound to be some.

On a more personal note, KFR again wishes to thank his wife Penny for her unwavering support, not only in his academic and tutorial work, but also in their joint efforts to convert time at the bridge table into ‘green points’ on their record. MPH is once more indebted to his wife, Becky, and his mother, Pat, for their tireless support and encouragement above and beyond the call of duty. MPH dedicates his contribution to this book to the memory of his father, Ronald Leonard Hobson, whose gentle kindness, patient understanding and unbreakable spirit made all things seem possible.

Ken Riley, Michael Hobson  
Cambridge, 2002
A knowledge of mathematical methods is important for an increasing number of university and college courses, particularly in physics, engineering and chemistry, but also in more general science. Students embarking on such courses come from diverse mathematical backgrounds, and their core knowledge varies considerably. We have therefore decided to write a textbook that assumes knowledge only of material that can be expected to be familiar to all the current generation of students starting physical science courses at university. In the United Kingdom this corresponds to the standard of Mathematics A-level, whereas in the United States the material assumed is that which would normally be covered at junior college.

Starting from this level, the first six chapters cover a collection of topics with which the reader may already be familiar, but which are here extended and applied to typical problems encountered by first-year university students. They are aimed at providing a common base of general techniques used in the development of the remaining chapters. Students who have had additional preparation, such as Further Mathematics at A-level, will find much of this material straightforward.

Following these opening chapters, the remainder of the book is intended to cover at least that mathematical material which an undergraduate in the physical sciences might encounter up to the end of his or her course. The book is also appropriate for those beginning graduate study with a mathematical content, and naturally much of the material forms parts of courses for mathematics students. Furthermore, the text should provide a useful reference for research workers.

The general aim of the book is to present a topic in three stages. The first stage is a qualitative introduction, wherever possible from a physical point of view. The second is a more formal presentation, although we have deliberately avoided strictly mathematical questions such as the existence of limits, uniform convergence, the interchanging of integration and summation orders, etc. on the
PREFACE TO THE FIRST EDITION

grounds that ‘this is the real world; it must behave reasonably’. Finally a worked example is presented, often drawn from familiar situations in physical science and engineering. These examples have generally been fully worked, since, in the authors’ experience, partially worked examples are unpopular with students. Only in a few cases, where trivial algebraic manipulation is involved, or where repetition of the main text would result, has an example been left as an exercise for the reader. Nevertheless, a number of exercises also appear at the end of each chapter, and these should give the reader ample opportunity to test his or her understanding. Hints and answers to these exercises are also provided.

With regard to the presentation of the mathematics, it has to be accepted that many equations (especially partial differential equations) can be written more compactly by using subscripts, e.g. $u_{xy}$ for a second partial derivative, instead of the more familiar $\partial^2 u/\partial x \partial y$, and that this certainly saves typographical space. However, for many students, the labour of mentally unpacking such equations is sufficiently great that it is not possible to think of an equation’s physical interpretation at the same time. Consequently, wherever possible we have decided to write out such expressions in their more obvious but longer form.

During the writing of this book we have received much help and encouragement from various colleagues at the Cavendish Laboratory, Clare College, Trinity Hall and Peterhouse. In particular, we would like to thank Peter Scheuer, whose comments and general enthusiasm proved invaluable in the early stages. For reading sections of the manuscript, for pointing out misprints and for numerous useful comments, we thank many of our students and colleagues at the University of Cambridge. We are especially grateful to Chris Doran, John Huber, Garth Leder, Tom Körner and, not least, Mike Stobbs, who, sadly, died before the book was completed. We also extend our thanks to the University of Cambridge and the Cavendish teaching staff, whose examination questions and lecture hand-outs have collectively provided the basis for some of the examples included. Of course, any errors and ambiguities remaining are entirely the responsibility of the authors, and we would be most grateful to have them brought to our attention.

We are indebted to Dave Green for a great deal of advice concerning typesetting in \LaTeX\ and to Andrew Lovatt for various other computing tips. Our thanks also go to Anja Visser and Graça Rocha for enduring many hours of (sometimes heated) debate. At Cambridge University Press, we are very grateful to our editor Adam Black for his help and patience and to Alison Woollatt for her expert typesetting of such a complicated text. We also thank our copy-editor Susan Parkinson for many useful suggestions that have undoubtedly improved the style of the book.

Finally, on a personal note, KFR wishes to thank his wife Penny, not only for a long and happy marriage, but also for her support and understanding during his recent illness – and when things have not gone too well at the bridge table! MPH is indebted both to Rebecca Morris and to his parents for their tireless
PREFACE TO THE FIRST EDITION

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Ken Riley, Michael Hobson and Stephen Bence
Cambridge, 1997