Strongly Elliptic Systems and Boundary Integral Equations

Partial differential equations provide mathematical models of many important problems in the physical sciences and engineering. This book treats one class of such equations, concentrating on methods involving the use of surface potentials. It provides the first detailed exposition of the mathematical theory of boundary integral equations of the first kind on non-smooth domains. Included are chapters on three specific examples: the Laplace equation, the Helmholtz equation and the equations of linear elasticity.

The book is designed to provide an ideal preparation for studying the modern research literature on boundary element methods.

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Frontmatter
More information

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Contents

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Frontmatter
More information

Preface

The study of integral equations in connection with elliptic boundary value problems has a long history, during which a variety of approaches has emerged. Rather than attempt a broad survey, I have chosen to pursue in detail just one approach, in which both the differential and integral formulations of a given boundary value problem are viewed abstractly as linear equations involving a bounded operator from a Hilbert space into its dual. The decisive property of this operator is that its associated sesquilinear form is positive and bounded below, apart perhaps from a compact perturbation.

In the classical Fredholm method, the solvability of the Dirichlet and Neumann problems is proved by reformulating them as integral equations of the second kind. Here, we effectively reverse this strategy, deriving key properties of the boundary integral equations from previously established results for the associated partial differential equations. Moreover, our approach leads to Fredholm integral equations of the *first* kind. The theory of such first-kind integral equations can be traced back to Gauss (see Chapter 1), and developed into the form presented here during the 1970s, in the work of Nedelec and Planchard [74], [76]; Le Roux [56], [57], [58]; and Hsiao and Wendland [42]. Those authors were all studying Galerkin boundary element methods, and although this book does not deal at all with numerical techniques, it is written very much from the perspective of a numerical analyst.

A major difficulty in a work such as this is the large amount of background material needed to present the main topics. Aware that readers differ in their prior knowledge, I have tried to adopt a middle path between, on the one hand, writing a textbook on functional analysis, distributions and function spaces, and on the other hand just stating, without proof or exposition, a litany of definitions and theorems. The result is that more than one-third of the text is made up of what might be considered technical preliminaries. My hope is that the book will be suitable for someone interested in finite or boundary element methods who wants a deeper understanding of the relevant non-numerical theory. I have aimed to keep the exposition as simple, concise and self-contained as possible, while at the same time avoiding assumptions that would be unrealistic for applications. Thus, I felt it essential to allow non-smooth domains, to consider systems and not just scalar equations, and to treat mixed boundary conditions.

Here is an outline of the contents.

Chapter 1 has two purposes. Firstly, it attempts to sketch the early history of the ideas from which the theory of this book developed. Secondly, it serves to introduce those ideas in an informal way, and to acquaint the reader with some of the notation used later.

The second chapter presents topics from linear functional analysis that are immediately relevant to what follows. I assume that the reader is already familiar with elementary facts about the topology of normed spaces, and of a few fundamental, deeper results such as the open mapping theorem and the Hahn–Banach theorem.

Chapter 3 develops the theory of Sobolev spaces on Lipschitz domains. After a quick treatment of distributions and Fourier transforms, we study in detail fractional- and negative-order spaces based on L_2. These spaces play an essential role in nearly all of the subsequent theory.

In Chapter 4, we begin our investigations of elliptic systems. A key tool is the first Green identity, used to arrive at the abstract (weak) formulation of a boundary value problem mentioned above. The centrepiece of the chapter is the Fredholm alternative for the mixed Dirichlet and Neumann problem on a bounded Lipschitz domain. We go on to prove some standard results on regularity of solutions, including the transmission property. The final section of the chapter proves some difficult estimates of Nečas [72] that relate the H^1-norm of the trace of a solution to the L_2-norm of its conormal derivative. These estimates are used later when showing that, even for general Lipschitz domains, the basic mapping properties of the surface potentials and boundary integral operators hold in a range of Sobolev spaces.

Chapter 5 is something of a technical digression on homogeneous distributions. As well as dealing with standard material such as the calculation of Fourier transforms, we include results from the thesis of Kieser [48], including the change-of-variables formula for finite-part integrals.

Chapters 6 and 7 form the heart of the book. Here, we study potentials and boundary integral operators associated with a strongly elliptic system of partial differential equations. Our overall approach is essentially that of Costabel [14], allowing us to handle Lipschitz domains. The first part of Chapter 6 deals with parametrices and fundamental solutions, and uses the results of Chapter 5. We then prove the third Green identity, and establish the main properties of the single- and double-layer potentials, including the familiar jump relations. Chapter 7 derives the boundary integral equations for the Dirichlet, Neumann and mixed problems, treating interior as well as exterior problems. The Fredholm alternative for the various boundary integral equations is established by showing positive-definiteness up to a compact perturbation, a property that is intimately related to the strong ellipticity of the associated partial differential operator.

Chapters 8–10 treat three of the simplest and most important examples of elliptic operators. For these specific cases, we can refine the general theory in certain respects. Chapter 8 deals with the Laplace equation, and includes a few classical topics such as spherical harmonics and capacity. Chapter 9 deals with the Helmholtz (or reduced wave) equation, and Chapter 10 gives a brief treatment of the linearised equilibrium equations for a homogeneous and isotropic elastic medium.

The book concludes with three appendices. The first of these proves Calderón's extension theorem for Sobolev spaces on Lipschitz domains, including the fractional-order case. The second gives a rapid but self-contained treatment of interpolation spaces and establishes the interpolation properties of Sobolev spaces on Lipschitz domains. The third proves a few facts about spherical harmonics.

At the end of each chapter and appendix is a set of exercises. These are of various types. Some are simple technical lemmas or routine calculations used at one or more points in the main text. Others present explicit solutions or examples, intended to help give a better feeling for the general theory. A few extend results in the text, or introduce related topics.

Some mention of what I have *not* covered also seems in order.

Many books treat Fredholm integral equations of the second kind. Well-known older texts include Kellogg [45] and Günter [35], and we also mention Smirnov [95] and Mikhlin [65, Chapter 18]. Problems on non-smooth domains are treated by Král [49] and Burago and Maz'ya [6], using methods from geometric measure theory, and by Verchota [102] and Kenig [46], [47] using harmonic analysis techniques. Works oriented towards numerical analysis include Kress [50], Hackbusch [36] and Atkinson [3]. Boundary value problems can also be reformulated as Cauchy singular integral equations, as in the pioneering work by Muskhelishvili [71]; for a modern approach, see Gohberg and Krupnik [28] or Mikhlin and Prößdorf [66].

Even for boundary integral equations of the first kind, the material presented in this book is by no means exhaustive. For instance, Costabel and Wendland [15] have generalised the approach used here to higher-order strongly elliptic equations. One can also study boundary integral equations as special cases of pseudodifferential equations; see, e.g., Chazarain and Piriou [10]. We make contact with the theory of pseudodifferential operators on several occasions, but do not attempt a systematic account of this topic. Other significant matters not treated include the L_p theory for $p \neq 2$, various alternative boundary conditions, especially non-linear ones, and a detailed study of the dominant singularities in a solution at corner points or edges of the domain.

During the period I have worked on this book, the Australian Research Council has provided support for a number of related research projects. I thank David Elliott for reading an early draft of the complete manuscript and making a number of helpful suggestions. I also thank Werner Ricker and Jan Brandts for the care with which they read through later versions of some of the chapters. Alan McIntosh and Marius Mitrea helped me negotiate relevant parts of the harmonic analysis literature. Visits to Mark Ainsworth at Leicester University, U.K., to Youngmok Jeon at Ajou University, Korea, and to the Mittag–Leffler Institute, Stockholm, provided valuable opportunities to work without the usual distractions, and made it possible for me to complete the book sooner than would otherwise have been the case. Needless to say, I am also indebted to many other people, who helped by suggesting references, discussing technical questions, and passing on their knowledge through seminars.

Sydney,
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