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Locality and the Hardy theorem

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But this conclusion [nonlocality] needs careful discussion in order to clarify what is going on. (Redhead 1987, p. 3)

Within the foundations of physics in recent years, Bell's theorem has played the role of what Thomas Kuhn calls a 'paradigm': that is, an exemplary piece of work that others learn from, imitate and develop. Following a period of articulation and consolidation, the first generation of developments of the Bell theorem was initiated by Heywood and Redhead (1983). They produced a nonlocality result in the algebraic style of the Bell–Kochen–Specker theorem (Bell 1966; Kochen and Specker 1967), moving away from the probabilistic relations characteristic of the Bell theorems proper. More recently a second generation develops results by Peres (1990), Greenberger–Horne–Zeilinger (1990), and Hardy (1993). In addition to moving away from probabilities, this generation tries to dispense with the limiting inequalities of the Bell theorem to yield so-called 'Bell theorems without inequalities'. With respect to probabilities, however, Hardy is a half-way house. It requires no inequalities but the result contradicts quantum mechanics under certain locality assumptions only if the statistical predictions of quantum mechanics hold in at least one case.

I want to examine the Hardy theorem and its interpretation. Initially, I intend to ignore respects in which it dispenses with probabilities because I want to point out the interesting significance of the theorem in a probabilistic context. We will see that when probabilities are restored, so are inequalities. Then we will see what the theorem has to contribute on the topic of locality.

1. Then Hardy example

According to Hardy, almost all the entangled states for a pair of systems give rise to a simple sort of Bell theorem. For our purposes a generic version of this will do.[1] So suppose we have a pair of systems whose state spaces are two dimensional (one can think of spin-1/2 systems, for instance), system I whose state space has orthonormal basis α, σ and system II with orthonormal basis β, τ. Denote by LC(...) a non-degenerate linear combination (that is, one with non-zero coefficients) of the enclosed terms. Let Ψ be the state of the combined (I + II) system. We will suppose that

$$\Psi = \mathrm{LC}(\alpha \otimes \tau, \sigma \otimes \beta, \sigma \otimes \tau). \tag{1a}$$

Collecting the terms, first in τ and then in σ, we get

$$\Psi = \mathrm{LC}(\alpha' \otimes \tau, \sigma \otimes \beta) \tag{1b}$$

and

$$\Psi = \mathrm{LC}(\alpha \otimes \tau, \sigma \otimes \beta') \tag{1c}$$

respectively, where $\alpha' = \mathrm{LC}(\alpha, \sigma)$ and $\beta' = \mathrm{LC}(\beta, \tau)$. Adding (1b) and (1c) and dividing by 2 yields

$$\Psi = \mathrm{LC}(\alpha' \otimes \tau, \sigma \otimes \beta', \sigma \otimes \beta, \alpha \otimes \tau). \tag{1d}$$

We can now read off various probabilistic statements from the form of these different representations of the joint state. For that purpose, let $A = P_{[\alpha]}$, $B = P_{[\beta]}$, $A' = I - P_{[\alpha']}$ and $B' = I - P_{[\beta']}$. Note that the relation between σ and A is not the same as that between α' and A'. Since there is no $\alpha \otimes \beta$ term in eqn 1a the result of projecting Ψ onto the $\alpha \otimes \beta$-space is null. Hence, where $P^{\Psi}(.)$ is the quantum probability in state Ψ, and writing $P^{\Psi}(AB)$ for $P^{\Psi}(A = 1 \ \& \ B = 1)$ – and so forth –

$$P^{\Psi}(AB) = 0. \tag{2a}$$

From eqn 1b, the result of projecting Ψ orthogonally to α' in the I-space leaves system II in state β, hence

$$P^{\Psi}(B|A') = 1. \tag{2b}$$

Similarly, from eqn 1c,

[1] The presentation below draws on Hardy's original (1993) and on the variant in Goldstein (1994). To map Hardy's discussion (U_i, D_i) onto my set U_1, $U_2 \leftrightarrow A$, B and D_1, $D_2 \leftrightarrow A'$, B'. The U_i are the same for Goldstein and his W_1, W_2 correspond (respectively) to my $I - A'$, $I - B'$.

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$$P^{\psi}(A|B') = 1. \tag{2c}$$

From eqn 1d, since neither $< \alpha|\alpha' >$ nor $< \beta|\beta' >$ is zero, the result of projecting Ψ so as to be orthogonal both to α' in the I-space and to β' in the II-space is not null. Hence,

$$P^{\Psi}(A'B') \neq 0. \tag{2d}$$

In eqn 1a the three-termed linear combination has non-zero coefficients the square of whose norms sum to 1, so $| < \Psi|\alpha \bigotimes \tau > |^2 + | < \Psi|\sigma \bigotimes \beta > |^2 \leq 1.$ That is, $P^{\Psi}(A) + P^{\Psi}(B) \leq 1.$ From eqn 2b, $P^{\Psi}(A') \leq P^{\Psi}(B)$. Adding $P^{\Psi}(A)$ to both sides yields

$$P^{\Psi}(A) + P^{\Psi}(A') \leq 1. \tag{2e}$$

Similarly from eqn 2c, $P^{\Psi}(B') \leq P^{\Psi}(A)$, which yields

$$P^{\Psi}(B) + P^{\Psi}(B') \leq 1. \tag{2f}$$

Extracting from the 0 to 1 probabilities, but using entirely similar reasoning about the geometry of the state space, the first three inferences of this series – eqns 2a, b, c – are the same as those drawn by Hardy. That is, talking about the values of quantities rather than probabilities for observing these values, he says that in state Ψ, $AB = 0$, that $(A' = 1) \Rightarrow (B = 1)$, and that $(B' = 1) \Rightarrow (A = 1)$. Hardy then draws the probabilistic conclusion 2d and urges that since local realism would sanction talk about locally possessed values it would lead instead to the conclusion that $A'B' = 0$. Hence he infers that a single A', B' measurement confirming the statistical prediction 2d would contradict local realism. Retaining the probabilities, the crux of the Hardy result would be a contradiction among eqns 2a, b, c, d that arises in the context of local hidden variables.

2. Random variables

The Bell inequalities have a purely probabilistic content as conditions governing whether a given set of probability distributions can be represented as the distributions of random variables. Given four pair distributions P_{AB}, $P_{A'B}$, $P_{AB'}$, $P_{A'B'}$ (say, on 0 and 1) with compatible singles P_A, P_B, $P_{A'}$, $P_{B'}$ (that is where the same marginal distribution P_A comes from P_{AB} and $P_{AB'}$, and so on) we can ask whether these fit together as the marginals of some four-distribution $P_{AA'BB'}$. Equivalently, we can ask whether there are random variables A, A', B and B' all defined on some common space whose single and joint distributions match the given singles and joints. Writing

$P(AB)$ for $P_{AB}(1,1)$ – and so forth – necessary and sufficient for this is the satisfaction of the generalized Bell inequalities:

$$-1 \leq P(A'B') + P(A'B) + P(AB') - P(AB) - P(A') - P(B') \leq 0. \quad \text{(GB)}$$

Interchanging first A with A', then B with B' and finally both together yields a total of eight inequalities, which constitute the required necessary and sufficient conditions (Fine 1982a, b).

The connection with local hidden variables (or 'local realism') is just that a typical EPR-type correlation experiment (Einstein, Podolsky & Rosen 1935) yields pair distributions and compatible singles for quantum observables as above, where A, A' are noncommuting observables defined in one wing of the experiment and B, B' in the other. A local hidden variables model provides a way of representing those observables as random variables over a common space. In particular, locality is what justifies saying – for example – that $A(x)$, the value of A at 'hidden state' x, is well defined without regard to other observables, their values or measurements. Similarly for the probabilities represented by the $P(.)$ distribution. Thus the random variables framework codifies the idea that in a given state Ψ there are determinate values for the observables that do not depend on distant measurements and that are distributed according to definite probabilistic laws. As I will suggest below (section 4), I believe that standard uses of this framework employ principles beyond locality. Still, it is a clean and perspicuous way of treating locality conditions, a way that disentangles the discussion from murky 'elements of reality' and potentially misleading counterfactual reasoning ('If instead of measuring A' we had measured A then … and also if instead of measuring B' we had measured B then, … .'). If we accept the standard framework, we can say that the (GB) inequalities provide the necessary and sufficient conditions for the existence of a local hidden variables model for a 2-by-2 EPR experiment.

Typically such an experiment works with imperfect correlations; that is, with joint probabilities neither 0 nor 1. We can ask, however, what would happen in the case where some one-way correlations are strict. Suppose, for instance, that measuring A' determined the outcome at B and that measuring B' determined the A outcome. That is, suppose, that the conditional probabilities $P(B|A')$ and $P(A|B')$ were both 1, as in the Hardy example. The following theorem characterizes this situation.

Theorem. If $P(B|A') = P(A|B') = 1$, then the pair distributions P_{AB}, $P_{A'B}$, $P_{AB'}$, $P_{A'B'}$, (on 0 and 1) with compatible singles P_A, P_B, $P_{A'}$, $P_{B'}$ are the distributions of random variables A, A', B and B' on some common space iff the following four conditions hold:

(i) $P(A'B') \leq P(AB)$
(ii) $P(A' \text{ or } B') \leq P(A \text{ or } B)$
(iii) $[P(A) - P(AB)] + [P(A') - P(A'B')] \leq 1$
(iv) $[P(B) - P(AB)] + [P(B') - P(A'B')] \leq 1.$

The theorem follows from the corresponding theorem for (GB) if we use conditions equivalent to $P(B|A') = P(A|B') = 1$; namely, that $P(A'B) = P(A')$ and $P(AB') = P(B')$. Indeed (i) is equivalent to the right side of (GB) under these assumptions. The Bell inequality $P(AB) + P(A'B) + P(AB') - P(A'B') \leq P(A) + P(B)$ is equivalent to (ii), that is, to $P(A') + P(B') - P(A'B') \leq P(A) + P(B) - P(AB)$. For (iii) the equivalent inequality is $P(A) + P(B') - 1 \leq P(AB)$ $+P(A'B')$ $+P(AB') - P(A'B)$. Interchanging A with A' and B with B' here yields, finally, the Bell inequality equivalent to (iv). Nothing new corresponds to the remaining four (GB) inequalities, each of which already follows from the stated conditions on the pair distributions.

We can now impose more structure, in particular the anti-correlations, $P(AB) = 0$, in the Hardy example.

Corollary 1. If $P(AB) = 0$, then the necessary and sufficient conditions reduce to

(a) $P(A'B') = 0$
(b) $\max[P(A) + P(A'), P(B) + P(B')] \leq 1$

Given that $P(AB) = 0$ and $0 \leq P(A'B')$, condition (i) holds iff (a) does. Similarly (b) is equivalent to (iii) and (iv). Finally by virtue of (a), (ii) automatically holds iff $P(AB) = 0$ since the assumptions of strict one-way correlation, that $P(A'B = P(A')$ and $P(AB') = P(B')$, imply that $P(A') \leq P(B)$ and that $P(B') \leq P(A)$.

Corollary 2. If $P(AB) = 0$ and $\max[P(A) + P(A'), P(B) + P(B')] \leq 1$, then

$$P(A'B') = 0 \text{ or, equivalently, } P(A'B') \leq P(AB)$$

is both necessary and sufficient for a random variables representation of the given joints and singles with strict correlations, $P(B|A') = P(A|B') = 1$.

3. The Hardy theorem

If we identify the quantum joints in the Hardy example with these distributions of random variables, all the requirements of corollary 2 are satisfied. From eqn 2a, $P(AB) = 0$. From eqn 2b, c, $P(B|A') = P(A|B') = 1$. From eqn 2e, f, $\max[P(A) + P(A'), P(B) + P(B')] \leq 1$. The condition $P(A'B') = 0$ that fails, according to eqn 2d, then, is precisely the condition whose satisfaction is both necessary and sufficient for a local hidden variables model of these probabilities. Thus for the Hardy case the equation $P(A'B') = 0$ (or the inequality $P(A'B') \leq P(AB)$) plays exactly the same role as do the (GB) inequalities for EPR in general. Although strict one-way correlations greatly simplify the reasoning, like Bell, Hardy has put his finger on precisely the central condition that makes local hidden variables possible for the case at hand.

Also, like the Bell theorem, the 'Hardy theorem' can be characterized as having two parts. One is a demonstration of a necessary condition for a hidden variables model. The second is the production of a generic example where that condition fails quantum mechanically. We can take

$$\text{if } P(B|A') = P(A|B') = 1 \text{ then } P(A'B') \leq P(AB)$$

for part (1). Then part (2) consists of the eqns 2b, c satisfying the 'if' clause and eqns 2a, d violating the consequent. Alternatively, we can take

$$\text{if } P(B|A') = P(A|B') = 1 \text{ and } P(AB) = 0, \text{ then } P(A'B') = 0$$

for part (1). Then eqns 2a, b, c satisfy the 'if' clause and eqn 2d violates the consequent.

We might call the first version the Hardy theorem 'with inequalities' and the second the Hardy theorem 'without inequalities'. Logically speaking, they are equivalent. Both versions identify the joint probabilities $P(..)$ that govern random variables with the quantum joint probabilities $P^{\Psi}(..)$ of the associated quantum observables. Indeed, it is that identification that makes part (2) of the theorem possible.

We can isolate what makes these versions equivalent. It is simply the probabilistic identity

$$P(A'B') = P(A'B'AB) \tag{3}$$

that must hold if all four variables are simultaneously representable. From eqn 3 it obviously follows that $P(A'B') \leq P(AB)$ and also that $P(A'B') = 0$

if $P(AB) = 0$. To see why eqn 3 holds note that in a random variables representation $P(A'B')$ is the marginal of a four-distribution as follows,

$$P(A'B') = P(A'B'AB) + P(A'B'\bar{A}B) + P(A'B'A\bar{B}) + P(A'B'\overline{AB}) \qquad (4)$$

(where the bar means that the variable underneath takes the value 0). Since $P(A|A') = P(A|B') = 1$ iff $P(\bar{A}B') = P(A'\bar{B}) = 0$, all but the first term of eqn 4 vanishes – to produce eqn 3.

4. Probabilities and locality

Fans of the Hardy theorem may not be happy with my presentation of that result, even though it highlights the importance of the contradictory condition picked out by Hardy. My version is probabilistic and it depends critically on probabilistic reasoning and inequalities and, moreover, on the identification of quantum joint probabilities with random variable joints. How much simpler just to state that if $(A' = 1) \Rightarrow (B = 1)$ and $(B' = 1) \Rightarrow (A = 1)$, then if $AB = 0$, so too $A'B' = 0$. This is simpler, to be sure, in terms of reasoning to the conclusion, but not in terms of interpreting the quantum theory. For, like my presentation, the quantum theory is also probabilistic and to move from those probabilities to statements about values of quantities requires imposing an interpretive structure. Since the 'simple' inference does not hold in the quantum mechanical Hardy example, clearly his interpretative structure goes beyond the usual reading of the quantum probabilities in terms of likely outcomes of measurements. Of course it is supposed to do just that since the interpretation is supposed to require locality, in order to contradict it. We shall see, however, that it does more.

We can see exactly what the reading in terms of possessed values does require; namely, the principle that where the quantum joint probability is zero, as in eqn 2a, the observables in question do not both take the 0-probability values; that is, that either $A \neq 1$ or $B \neq 1$ in the case of eqn 2a. While this may seem like a harmless and modest principle, it is not. Years ago I showed that this principle is equivalent to the general Kochen–Specker functional condition: $f(Q)(x) = f[Q(x)]$ (Fine 1974). The connection is easy to see, for in any state Ψ and for any observable Q, $P^{\Psi}(Q = q \,\&\, f(Q) \neq f(q)) = 0$. So, the principle implies that $Q(x) = q$ only if $f(Q)(x) = f(q)$; that is, that $f(Q)(x) = f[Q(x)]$. This general functional condition, in turn, is equivalent to the product and sum rules: that the value possessed by the product of two observables (or their sum) is just obtained by multiplying (respectively, adding) the values possessed by the individual observables.[2] Even apart from locality considerations these rules are already inconsistent with the quantum theory. So the seemingly modest interpretive principle that governs the 'simple' form of the Hardy theorem is not at all harmless; indeed it is inconsistent with the quantum theory. The inconsistency runs deep and derives from how the basic framework of random variables is used.

The principle

$$\text{where } P^{\Psi}(A = q \;\&\; B = r) = 0, \text{ then } A \neq q \text{ or } B \neq r \tag{5a}$$

is in fact equivalent to the assumption about the random variables framework used in the two probabilistic versions of the Hardy theorem in section 3, that the joint probabilities of that framework match the quantum joints for observables where the latter are defined.[3] If it is locality that concerns us, this is not an assumption we need to make. For we can use the locality automatically built into the random variables framework – that values and probabilities are determinate and independent of distant measurements – and still entirely avoid this assumption on joint probabilities. Here is how. In a given state Ψ, make the usual association for a hidden variables construction between quantum observables and random variables. For each single observable this returns the quantum probabilities as the distribution of the associated random variable. To get joint probabilities where they are defined quantum mechanically (that is, for commuting observables) *do not* go to the (well-defined) joint distributions of the associated random variables. That would just re-instate the above principle. Instead, use the quantum mechanical identity

$$P^{\Psi}(A = q \;\&\; B = r) = P^{\Psi}(\chi_q(A)\chi_r(B) = 1) \tag{5b}$$

where $\chi(.)$ is the characteristic function (that is, $\chi_q(x) = 1$ for $x = q$ and 0 otherwise). The product observable $\chi_q(A)\chi_r(B)$ will correspond to some random variable, say C, whose distribution is quantum mechanical. If we assign the quantum joint probabilities by identifying $P(C = 1)$ with the right

[2] The equivalence also requires the rule ('spectrum rule') that the only possible values assigned to an observable in a state are those with non-zero probability in that state. I assume this in the discussions below.

[3] Fine (1974, pp. 261–4) shows the equivalence between the product rule and this assumption on joint distributions, given the spectrum rule of note 2. The claim in the text follows from that.

side of eqn 5b and then reading eqn 5b from right to left, they will be correct.

The preceding construction is neither pretty nor simple. But it shows something. It shows that the Hardy theorem, whether in the probabilistic versions explored in section 3 or in the apparently simpler version about possessed values, depends on more than locality. It depends, in addition, on a tacit requirement for how to deploy the framework of random variables in building a hidden variables model: namely, that we should employ the joint distribution structure of the random variables and demand that where applicable it match the quantum joints. If you think about it, however, this requirement is not really compelling, since we know in advance that the match can at best be partial. For all pairs of random variables have joint distribution but only some pairs of the quantum observable do. Thus any deployment of random variables necessarily involves excess structure.

The preceding construction shows something else too. It shows how to build a local hidden variables model for the Hardy example. All we need do is to make the suggested construction in the given state Ψ for the observables A, A', B, B', AB, AB', $A'B$, $A'B'$, $I-A$ and $I-B$, $A'(I-B)$ and $(I-A)B$. This will give a possessed value to each observable and probabilities that match those of quantum mechanics, as in eqn 2. We may find, for some 'hidden states', that $A'=1$ and $B'=1$, that $A=1$ but that $B=0$ and $(I-B)=1$, even though eqn 2b – $P^{\Psi}(B|A')=1$– holds. For eqn 2b is equivalent to $P^{\Psi}(A'\bar{B})=0=P^{\Psi}[A'(I-B)]$. The right side of course requires that the product $[A'(I-B)]=0$ but this is now compatible with $A'=1$ and $(I-B)=1$, since we no longer insist on eqn 5a and hence on the product rule. I said it was not pretty. But it is local, respects the probabilities of quantum mechanics and can accommodate all the measurement results. The 'funny' values – the ones that violate the product rule, or the like – can be regarded as values assigned to observables that do not commute with the ones being measured. From the quantum mechanical point of view these values are truly hidden, but they are locally assigned nevertheless.

The counterfactual reasoning that usually supports the interpretation of the Hardy theorem as a nonlocality proof goes wrong at the very start. Even before it gets entangled in nested counterfactuals it assumes that if a measurement turns up $A'=1$, then by virtue of $P^{\Psi}(B|A')=1$ we must have that $B=1$. We see above, however, that locality alone does not make this necessary; that is, not unless B is co-measured with A', in which case it will follow (just as it does in quantum mechanics). Reference to 'elements of reality' here – with its historically misleading echoes of EPR[4] – is equally off the track. For the quantum probability assignment $P^{\Psi}(B|A') = 1$ does not say that we can predict the value $B = 1$ from any measurement yielding $A' = 1$ that does not disturb the B system. It says that if we measure A' and B *together*, then where we find that $A' = 1$ we also find that $B = 1$. So even if we follow the prescription for elements of reality, all we can say is that where we measure A' and B together and find $A' = 1$ we can assign a $B = 1$ 'element of reality'. The Hardy argument, however, begins by supposing we have measured A' with B', not with B. In this situation there is no $B = 1$ 'element of reality' at all.

Beneath all these sophisticated arguments, I suggest, is a very simple conception for how we 'should' assign values when we try to respect locality and how we 'should' make that match up with the quantum probabilities. That conception is the random variables framework with the assumption of eqn 5a, or some equivalent. My point is that this conception goes well beyond the commitments of locality, which can be salvaged by assigning values differently. That means that the Hardy theorem, like other variants on Bell, is not a 'proof of nonlocality'. It is a proof that locality cannot be married to the assignment of determinate values in the recommended way. That is interesting and significant. It is not, however, a demonstration that quantum mechanics is nonlocal, much less (as some proclaim) that nature is.

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[4] I am sure that the interest in 'elements of reality' derives from their association with Einstein (the 'E' of EPR). The 'reality criterion' that governs the introduction of these elements in EPR, however, is almost certainly due to Podolsky, who wrote the paper. In EPR that criterion plays a subtle and minor role, quite different from its use in the Hardy literature and other recent writings on Bell's theorem without inequalities. That use, rather, is a descendant of the apocryphal version of EPR made up by Bohr (1935). See Beller & Fine (1993) for details.