GEOMETRY OF SETS AND MEASURES IN
EUCLIDEAN SPACES

Fractals and rectifiability

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To Vappu
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Basic notation

We introduce here the notation for some basic concepts which are not defined in the text. A more extensive glossary of notation is given at the end of the book.

\( \mathbb{Z} \), the set of integers.
\( \mathbb{R} \), the set of real numbers.
\( \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} \).
\( \mathbb{C} \), the set of complex numbers.
\( \overline{z} \), Re \( z \) and Im \( z \) are the complex conjugate, real part and imaginary part of \( z \in \mathbb{C} \).
\( \mathbb{R}^n \), the \( n \)-dimensional euclidean space equipped with the inner product \( x \cdot y \) and norm \( |x| \).
\( S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \), the unit sphere.
\( [a,b] \), \( (a,b) \), \( [a,b) \) and \( (a,b] \) are the closed, open and half-open intervals in \( \overline{\mathbb{R}} \) with end-points \( a,b \in \overline{\mathbb{R}} \).
\( \mathcal{L}^n \), the Lebesgue measure on \( \mathbb{R}^n \).
\( \alpha(n) = \mathcal{L}^n\{ x \in \mathbb{R}^n : |x| \leq 1 \} \), the volume of the unit ball.
\( \overline{A} = \text{Cl} \, A \), the closure of the set \( A \).
\( \partial A \), the boundary of \( A \).
\( \chi_A \), the characteristic function of \( A \).
\( A + B = \{ x + y : x \in A, \ y \in B \} \).
\( A + a = \{ x + a : x \in A \} \).
\( \text{card} \, A \), the number points in the set \( A \); possibly 0 or \( \infty \).
\( \bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A \), the union of the set family \( \mathcal{A} \).
\( \bigcap \mathcal{A} \), the intersection of \( \mathcal{A} \).

We often use notation like \( \{ x : f'(x) > 0 \} \) to mean the set of those points \( x \) where the derivative \( f'(x) \) exists and is positive.

The symbol \( \Box \) denotes the end of the proof.