1 INTRODUCTION AND MATHEMATICAL PRELIMINARIES

Summary. In this chapter we outline some of the topics treated in later chapters. We then review a number of basic mathematical results required from time to time in the theoretical development of subsequent chapters: summation notation; factorial and combinatorial notation; power notation; differentiation to find the slope of a curve; maxima and minima; and the exponential and natural logarithmic functions. It must be emphasised, however, that the reader does not have to have a detailed knowledge of these results to be able to appreciate the statistical theory of insurance as described in the remainder of this book.

1.1 Introduction

The title of this book is *Introductory statistics with applications in general insurance*. It is introductory in the sense that it presents fundamental statistical ideas and explains their application in general insurance. It does not give the last word on any of the topics treated, but leaves it to the reader to pursue in depth, through the references given at the end of each section, those matters which interest him or her most.

The early chapters form a general introduction to the study of probability and statistics. A chapter on statistical distributions, useful in general insurance, follows and thereafter the emphasis is on applications – inferences from general insurance data; exposure and the estimation of claim frequency rates; calculation of risk premium and risk premium for excess of loss reinsurance; experience rating and credibility; no claim discount systems; simulation of general insurance problems; methods for estimating outstanding claim provisions; risk theory and its application to retention levels.

Quantitative methods always involve mathematical formulae and computation, and statistics is no exception. What is remarkable, however, is the extent to which problems in an area as complex as general insurance can be analysed and solved using little more than basic school mathematics.

The remainder of this chapter is devoted to a review of selected mathematical topics required in subsequent chapters. The main purpose is to refresh the memory of the reader who has been away from mathematical studies longer than he or she cares to remember. The reader familiar with all of these topics should move directly to chapter 2.

1.2 Summation notation

In statistical work, we often need to calculate the sum of a number of quantities $x_1, x_2, \ldots x_n$. We can of course write the sum out longhand as

$$x_1 + x_2 + x_3 + \ldots + x_n. \tag{1.2.1}$$

A standard shorthand notation has been developed, however, which makes use of the upper case Greek letter sigma (for sum):

$$\sum_{i=1}^{n} x_i \tag{1.2.2}$$

In essence, this formula says 'sum all the x_i values from $i = 1$ to $i = n$'. Thus

■ $$\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n. \tag{1.2.3}$$

Often, an alternative 'dummy' subscript will be used instead of i.

Example 1.2.1. Evaluate $\Sigma_{i=1}^{5} i^2$.

We need to sum all the values of i^2 from $i = 1$ to $i = 5$. In other words,

$$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

Example 1.2.2. Evaluate $\Sigma_{r=1}^{4} x_r^2$ where $x_1 = 1.2$ $x_2 = 1.3, x_3 = 1.5,$ $x_4 = 1.8$.

We need to sum all the values of x_r^2 from $r = 1$ to $r = 4$. In other words,

$$\sum_{r=1}^{4} x_r^2 = (1.2)^2 + (1.3)^2 + (1.5)^2 + (1.8)^2 = 8.62.$$

Further reading: Johnson & Bhattacharya [14] 640–2; Stein & Barcellos [22] 256–60.

1.3 Factorial notation $n!$

The product of the first n natural numbers is frequently required in mathematical and statistical work. A convenient shorthand has therefore been devised for this product and is referred to as *factorial n*:

■ $$n! = 1 \times 2 \times 3 \times \ldots \times n. \tag{1.3.1}$$

By convention 0! is set equal to 1. Clearly, factorial n satisfies the recurrence equation

■ $$n! = n \times (n-1)! \tag{1.3.2}$$

Using this recurrence equation, we see that the factorials of 0, 1, 2, 3, 4, 5 and 6 are, respectively, 1, 1, 2, 6, 24, 120 and 720.

Further reading: Freund [6] 6; Stein & Barcellos [22] S23.

1.4 Combinatorial notation $\binom{n}{r}$

A committee consists of five persons. How many ways can we select a sub-committee of two from among them?

Let us denote the members of the committee by the letters *A, B, C, D* and *E*. The possible sub-committees are then as follows:

AB AC AD AE BC
BD BE CD CE DE .

Ten different sub-committees are possible.

Generally, the number of different ways of selecting r objects from n is denoted[1] by $\binom{n}{r}$ and

■ $$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{n\,(n-1)\ldots(n-r+1)}{1 \times 2 \times \ldots \times r} . \tag{1.4.1}$$

In our numerical example $n = 5$, $r = 2$, and

$$\binom{5}{2} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1) \times (3 \times 2 \times 1)} = \frac{5 \times 4}{1 \times 2} = 10 .$$

Situations also arise where we need to evaluate $[x\,(x-1)\ldots(x-r+1)]/r!$ for non-integer values of x. The definition of $\binom{n}{r}$ can be generalised to cover all possible values of x (integer, non-integer, positive, or negative) as follows:

■ $$\binom{x}{r} = \frac{x\,(x-1)\ldots(x-r+1)}{1 \times 2 \times \ldots \times r} . \tag{1.4.2}$$

This formula, however, only has a combinatorial meaning when x is a non-negative integer.

Example 1.4.1. Evaluate $\binom{n}{r}$ for all possible values of r and values of n from 0 to 7 inclusive. Use formula (1.4.1).

[1] Some textbooks denote the number of different ways of selecting r objects from n by ${}^nC_r$.

Sample calculations are as follows:

$$\binom{0}{0} = \frac{0!}{0!\,0!} = \frac{1}{1 \times 1} = 1 \ ;$$

$$\binom{1}{0} = \frac{1!}{0!\,1!} = \frac{1}{1 \times 1} = 1 \ ;$$

$$\binom{1}{1} = \frac{1!}{1!\,0!} = \frac{1}{1 \times 1} = 1 \ ;$$

$$\binom{2}{0} = \frac{2!}{0!\,2!} = \frac{2}{1 \times 2} = 1 \ ;$$

$$\binom{2}{1} = \frac{2!}{1!\,1!} = \frac{2}{1 \times 1} = 2 \ ;$$

$$\binom{2}{2} = \frac{2!}{2!\,0!} = \frac{2}{2 \times 1} = 1 \ ;$$

$$\binom{6}{3} = \frac{6!}{3!\,3!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (3 \times 2 \times 1)} = 20 \ .$$

The full set of results is shown as table 1.4.1 in the form of a *Pascal triangle.* Note that each entry is equal to the sum of two other entries: the entry immediately above, and the entry above but one place to the left.

Table 1.4.1. *The Pascal triangle – a table of values of* $\binom{n}{r}$

n \ r	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

Example 1.4.2. Evaluate $\binom{1.5}{3}$.

$$\binom{1.5}{3} = \frac{1.5 \times (1.5-1) \times (1.5-2)}{1 \times 2 \times 3} = \frac{1.5 \times 0.5 \times (-0.5)}{1 \times 2 \times 3} = -0.0625 \ .$$

Further reading: Freund [6] 1–14; Stein & Barcellos [22] S22–S24.

1.5 Power notation

The sixth power of a number x is x^6, and

$$\begin{aligned} x^6 &= x \times x \times x \times x \times x \times x \\ &= (x \times x \times x \times x) \times (x \times x) \\ &= (x \times x) \times (x \times x) \times (x \times x) \\ &= (x \times x \times x) \times (x \times x \times x) . \end{aligned} \tag{1.5.1}$$

We see, therefore, that

$$x^6 = x^4 \times x^2 = (x^2)^3 = (x^3)^2, \tag{1.5.2}$$

and

$$x^6/x^4 = x^2 . \tag{1.5.3}$$

These equations are of course particular examples of the following general relationships connecting powers:

- $x^a \times x^b = x^{a+b}$; (1.5.4)
- $x^a/x^b = x^{a-b}$; (1.5.5)
- $(x^a)^b = x^{ab} = (x^b)^a$. (1.5.6)

It is clear that these relationships hold whenever a, b and $a - b$ are positive integers. They are, in fact, valid for *all a* and *b,* positive or negative, integral or fractional, provided the following conventions are adopted:

- $x^0 = 1$; (1.5.7)
- $x^{-a} = 1/x^a$; (1.5.8)
- $x^{\frac{1}{a}}$ = ath root of x. (1.5.9)

Modern pocket calculators often have x^y keys which allow the rapid calculation of any power y (positive, negative or zero; fractional or integral) of any positive number x. If such a key is not available, the following relationship can be used:[2]

- $x^y = e^{y \ln x}$. (1.5.10)

Example 1.5.1. Evaluate
(a) 2^{-3} .
(b) $3^{\frac{1}{2}}$.

[2] The exponential function e^x and the natural logarithmic function $\ln x$ are described below in sections 1.9 and 1.10 respectively.

(c) $3^{-\frac{1}{2}}$.
(d) $27^{\frac{2}{3}}$.
(e) $27^{-\frac{2}{3}}$.
(f) $(5^2)^{\frac{1}{2}} \times 5^2 \times 5^{-3}$.

Solution:
(a) $2^{-3} = \left(\frac{1}{2}\right)^3 = \frac{1}{8} = 0.125$.
(b) $3^{\frac{1}{2}} = \sqrt{3} = 1.7321$.
(c) $3^{-\frac{1}{2}} = \left(\frac{1}{3}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}} = 0.577\ 35$.
(d) $(27)^{\frac{2}{3}} = [(27)^{\frac{1}{3}}]^2 = 3^2 = 9$.
(e) $(27)^{-\frac{2}{3}} = 1/(27)^{\frac{2}{3}} = \frac{1}{9} = 0.111\ 11$.
(f) $(5^2)^{\frac{1}{2}} \times 5^2 \times 5^{-3} = 5^1 \times 5^2 \times 5^{-3} = 5^0 = 1$.

Example 1.5.2. Evaluate $(10.8673)^{0.45}$.

The evaluation may be performed directly using the x^y key on a pocket calculator. The answer is 2.925 87.

Alternatively, we can use (1.5.10):

$$x = 10.8673\ ;$$
$$\ln x = 2.385\ 758\ ;$$
$$0.45 \ln x = 1.073\ 591\ 1\ ;$$
$$(10.8673)^{0.45} = e^{0.45 \ln x} = 2.925\ 87\ .$$

A calculator with an x^y key actually follows this procedure automatically.

Further reading: Gillet [7]13, 44–8, 52–4, 97–104, 154–6; Hughes-Hallett [9] 43; Stein & Barcellos [22] S27–S32.

1.6 Differentiation; the slope of a curve

Fig. 1.6.1 shows a straight line which rises with increasing values of x. When $x = 2$, the height of the line is 2.0 and when $x = 4$, the height is 3.0. It is clear, therefore, that as we move along horizontally 2 units, the line rises vertically 1 unit. We may say that the line has a slope of 1 in 2 or 0.5.

Alternatively, we may note that between $x = 2$ and $x = 5$, the line rises 1.5 units from 2 to 3.5. The slope is, therefore, 1.5 in 3 or 0.5. The answer is the same as that obtained previously, because the slope of a straight line is constant. It does not become more steep; nor does it become less steep.

A line *falling* 1 unit between $x = 2$ and $x = 4$ would have a slope of -1 in 2 or -0.5.

Fig. 1.6.2 shows a curve which is initially fairly flat, but which becomes progressively steeper and steeper. An estimate of the slope of the curve at the point x may be obtained as follows.

1. Note the height $f(x)$ of the curve at the point x.
2. Move along the curve to the right and note the height $f(x+H)$ at the point $x+H$.

The rise over horizontal distance H is $f(x+H)-f(x)$, and we deduce that the slope at the point x is approximately $f(x+H)-f(x)$ in H or

$$\frac{f(x+H)-f(x)}{H} . \tag{1.6.1}$$

Fig. 1.6.1. A straight line.

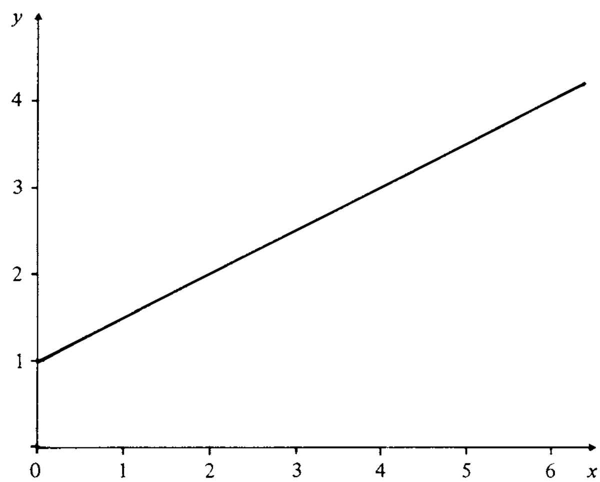
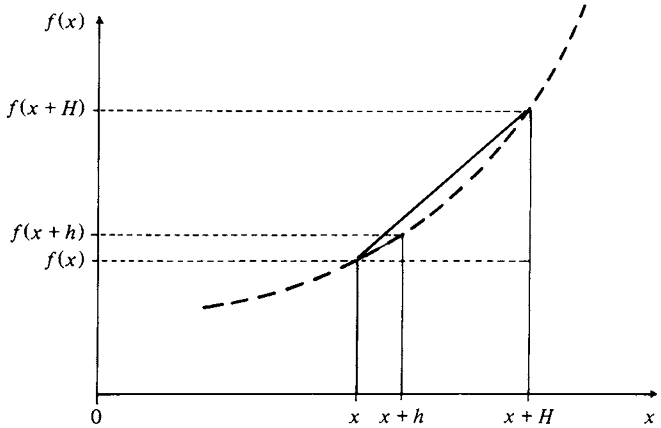


Fig. 1.6.2. Estimating the slope of a curve.



The curve, however, is rising more and more steeply, and it is apparent that if we select a large value of H, a poor estimate of the slope at the point x will be obtained. A better estimate will result from using a smaller horizontal distance h (fig. 1.6.2). The best estimate is obtained by making h as small as possible (i.e. close to zero) and calculating the limit, as h gets closer and closer to zero, of $[f(x+h) - f(x)]/h$. The slope we have defined in this manner is often referred to as the *derivative* of the $f(x)$ curve at the point x and denoted by $\mathrm{d}f(x)/\mathrm{d}x$ or $\mathrm{d}f/\mathrm{d}x$. We may write

■ $$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad . \qquad (1.6.2)$$

This formula is perfectly general. Let us now consider a special case and show how the formula may be applied in practice. Let us imagine that the height $f(x)$ of a curve at the point x is equal to x^3. At the point $x + h$, the height will be $(x + h)^3$. The curve rises, therefore, $(x + h)^3 - x^3$ in horizontal distance h.

We note that $(x + h)^3 = x^3 + 3hx^2 + 3h^2x + h^3$. It follows that

$$\frac{1}{h}\,[f(x+h) - f(x)] = \frac{1}{h}\,[(x^3 + 3hx^2 + 3h^2x + h^3) - x^3]$$

$$= \frac{1}{h}\,[3hx^2 + 3h^2x + h^3]$$

$$= 3x^2 + 3hx + h^2 \quad .$$

In the limit as h tends to zero, the last two terms become zero, and we deduce that the slope of the curve $f(x) = x^3$ at the point x is given by

$$\frac{\mathrm{d}}{\mathrm{d}x} x^3 = 3x^2 \quad . \qquad (1.6.3)$$

This is an example of a well-known general result

■ $$\frac{\mathrm{d}}{\mathrm{d}x} x^n = nx^{n-1} \quad , \qquad (1.6.4)$$

which is valid for all values of the constant n (integral, fractional, positive or negative). In fact, the derivatives of most standard mathematical functions are well known and do not need to be derived from first principles each time they are required.

Three other results, which are readily proven, need to be noted (A is a fixed constant independent of x):

■ $$\frac{\mathrm{d}}{\mathrm{d}x} A = 0 \; ; \qquad (1.6.5)$$

- $$\frac{d}{dx} Af(x) = A \frac{d}{dx} f(x); \qquad (1.6.6)$$
- $$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x). \qquad (1.6.7)$$

Sometimes the slope $d(df/dx)/dx$ of the slope curve df/dx is required. This *second derivative* is usually denoted by d^2f/dx^2. Third and higher derivatives are also encountered.

Example 1.6.1. What is the function $F(x)$ which when differentiated is equal to $3x^2$?

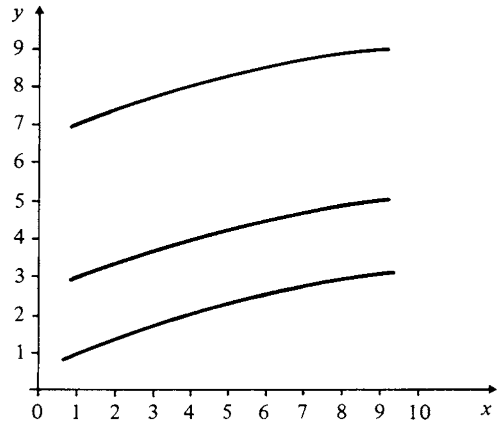
We might be inclined (in view of (1.6.3)) to answer immediately: x^3. It is indeed true that the derivative of x^3 is $3x^2$, but so is the derivative of $x^3 + 37$ and the derivative of $x^3 + A$, for any constant A whatsoever, because the derivative of a constant A is zero (equation (1.6.5)) and the derivative of a sum is the sum of the derivatives (equation (1.6.7)).

The function $F(x)$ which when differentiated yields $3x^2$ is

$$F(x) = x^3 + A \qquad (1.6.8)$$

and the constant A cannot be determined without further information. This may at first seem surprising. It is not really so surprising when we observe that, by knowing the slope of a line at all points on it, we can work out the relative heights of various points on the line, but we cannot determine the absolute height of any point on it without knowing the absolute height of some base point on that line. The three curves in fig. 1.6.3 have the same slopes as each other, but they are located at different altitudes.

Fig. 1.6.3. Three curves with the same slopes.



Example 1.6.2. Find the first, second and third derivatives of the function $f(x) = x^5$ at the point $x = 2$.

According to (1.6.4) the first derivative or slope of the curve $f(x) = x^5$ at the point x is

$$\frac{df}{dx} = 5x^4 .$$

At the point $x = 2$, therefore, the first derivative or slope is $5 \times 2^4 = 80$.

Using (1.6.4) again and (1.6.6) as well, we see that the second derivative, or slope of the slope curve at the point x is

$$\frac{d^2f}{dx^2} = \frac{d}{dx}(5x^4) = 20x^3 .$$

At the point $x = 2$, therefore, the second derivative is $20 \times 2^3 = 160$.

In a similar manner, the third derivative, or slope of the slope of the slope curve of $f(x)$ at the point x is

$$\frac{d^3f}{dx^3} = \frac{d}{dx}(20x^3) = 60x^2 .$$

At the point $x = 2$, therefore, the third derivative is $60 \times 2^2 = 240$.

Further reading: Hughes-Hallett [9] 120–32, 139, 143–4, 155; Salas & Hille [21] 10–11, 103–8; Stein & Barcellos [22] 104–6, 113–21, S12–S18.

1.7 Maxima and minima

Fig. 1.7.1 depicts a curve $f(x)$ which rises to a maximum and then falls away. The slope of the curve is positive before the maximum and negative afterwards. At the maximum itself, the slope is zero. In other words,

■ $$\frac{df}{dx} = 0 \qquad (1.7.1)$$

at a maximum point on a curve $f(x)$.

The slope of the curve is quite steep well before the maximum but becomes flatter and flatter as the maximum is approached. After passing the maximum, the slope downwards becomes steeper and steeper. Mathematically, therefore, we see that the derivative df/dx changes from being positive prior to the maximum to negative after the maximum (fig. 1.7.2). The slope of the slope curve at the maximum of $f(x)$ must be negative. In other words, the second derivative (section 1.6).

■ $$\frac{d^2f}{dx^2} < 0 \qquad (1.7.2)$$

at a maximum point on the $f(x)$ curve.