1

Vector and tensor analysis

Vectors and scalars

Vector methods have become standard tools for the physicists. In this chapter we discuss the properties of the vectors and vector fields that occur in classical physics. We will do so in a way, and in a notation, that leads to the formation of abstract linear vector spaces in Chapter 5.

A physical quantity that is completely specified, in appropriate units, by a single number (called its magnitude) such as volume, mass, and temperature is called a scalar. Scalar quantities are treated as ordinary real numbers. They obey all the regular rules of algebraic addition, subtraction, multiplication, division, and so on.

There are also physical quantities which require a magnitude and a direction for their complete specification. These are called vectors *if* their combination with each other is commutative (that is the order of addition may be changed without affecting the result). Thus not all quantities possessing magnitude and direction are vectors. Angular displacement, for example, may be characterised by magnitude and direction but is not a vector, for the addition of two or more angular displacements is not, in general, commutative (Fig. 1.1).

In print, we shall denote vectors by boldface letters (such as **A**) and use ordinary italic letters (such as A) for their magnitudes; in writing, vectors are usually represented by a letter with an arrow above it such as $\vec{A}$. A given vector **A** (or $\vec{A}$) can be written as

$$\mathbf{A} = A\hat{A}, \tag{1.1}$$

where A is the magnitude of vector **A** and so it has unit and dimension, and $\hat{A}$ is a dimensionless unit vector with a unity magnitude having the direction of **A**. Thus $\hat{A} = \mathbf{A}/A$.

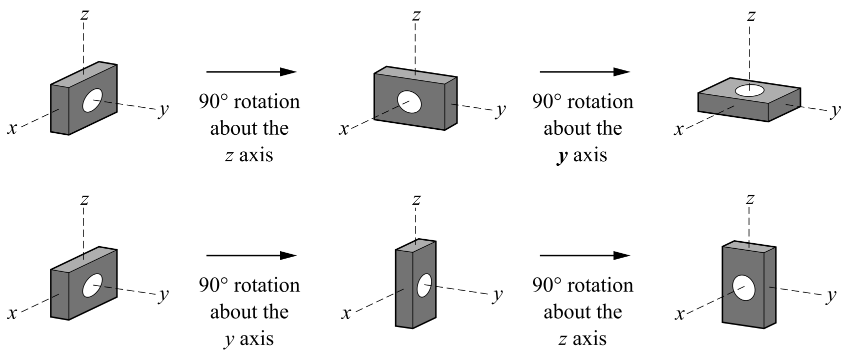


Figure 1.1. Rotation of a parallelpiped about coordinate axes.

A vector quantity may be represented graphically by an arrow-tipped line segment. The length of the arrow represents the magnitude of the vector, and the direction of the arrow is that of the vector, as shown in Fig. 1.2. Alternatively, a vector can be specified by its components (projections along the coordinate axes) and the unit vectors along the coordinate axes (Fig. 1.3):

$$\mathbf{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A\hat{e}_3 = \sum_{i=1}^{3} A_i\hat{e}_i, \tag{1.2}$$

where $\hat{e}_i$ ($i = 1, 2, 3$) are unit vectors along the rectangular axes x_i ($x_1 = x$, $x_2 = y$, $x_3 = z$); they are normally written as $\hat{i}$, $\hat{j}$, $\hat{k}$ in general physics textbooks. The component triplet (A_1, A_2, A_3) is also often used as an alternate designation for vector $\mathbf{A}$:

$$\mathbf{A} = (A_1, A_2, A_3). \tag{1.2a}$$

This algebraic notation of a vector can be extended (or generalized) to spaces of dimension greater than three, where an ordered n-tuple of real numbers, $(A_1, A_2, \ldots, A_n)$, represents a vector. Even though we cannot construct physical vectors for $n > 3$, we can retain the geometrical language for these n-dimensional generalizations. Such abstract "vectors" will be the subject of Chapter 5.

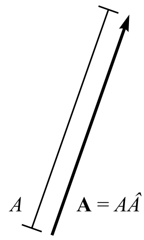


Figure 1.2. Graphical representation of vector **A**.

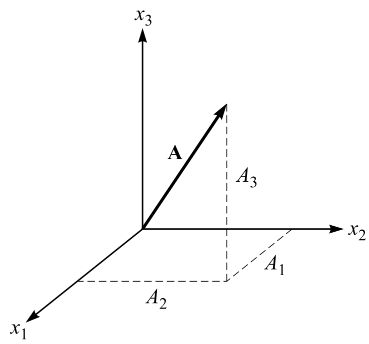


Figure 1.3. A vector **A** in Cartesian coordinates.

Direction angles and direction cosines

We can express the unit vector $\hat{A}$ in terms of the unit coordinate vectors $\hat{e}_i$. From Eq. (1.2), $\mathbf{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A\hat{e}_3$, we have

$$\mathbf{A} = A\left(\frac{A_1}{A}\hat{e}_1 + \frac{A_2}{A}\hat{e}_2 + \frac{A_3}{A}\hat{e}_3\right) = A\hat{A}.$$

Now $A_1/A = \cos\alpha$, $A_2/A = \cos\beta$, and $A_3/A = \cos\gamma$ are the direction cosines of the vector **A**, and α, β, and γ are the direction angles (Fig. 1.4). Thus we can write

$$\mathbf{A} = A(\cos\alpha\hat{e}_1 + \cos\beta\hat{e}_2 + \cos\gamma\hat{e}_3) = A\hat{A};$$

it follows that

$$\hat{A} = (\cos\alpha\hat{e}_1 + \cos\beta\hat{e}_2 + \cos\gamma\hat{e}_3) = (\cos\alpha, \cos\beta, \cos\gamma). \quad (1.3)$$

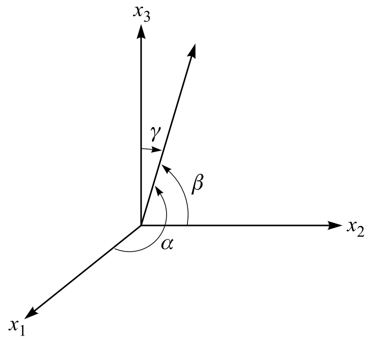


Figure 1.4. Direction angles of vector **A**.

Vector algebra

Equality of vectors

Two vectors, say **A** and **B**, are equal if, and only if, their respective components are equal:

$$\mathbf{A} = \mathbf{B} \quad \text{or} \quad (A_1, A_2, A_3) = (B_1, B_2, B_3)$$

is equivalent to the three equations

$$A_1 = B_1, A_2 = B_2, A_3 = B_3.$$

Geometrically, equal vectors are parallel and have the same length, but do not necessarily have the same position.

Vector addition

The addition of two vectors is defined by the equation

$$\mathbf{A} + \mathbf{B} = (A_1, A_2, A_3) + (B_1, B_2, B_3) = (A_1 + B_1, A_2 + B_2, A_3 + B_3).$$

That is, the sum of two vectors is a vector whose components are sums of the components of the two given vectors.

We can add two non-parallel vectors by graphical method as shown in Fig. 1.5. To add vector **B** to vector **A**, shift **B** parallel to itself until its tail is at the head of **A**. The vector sum **A** + **B** is a vector **C** drawn from the tail of **A** to the head of **B**. The order in which the vectors are added does not affect the result.

Multiplication by a scalar

If c is scalar then

$$c\mathbf{A} = (cA_1, cA_2, cA_3).$$

Geometrically, the vector $c\mathbf{A}$ is parallel to **A** and is c times the length of **A**. When $c = -1$, the vector $-\mathbf{A}$ is one whose direction is the reverse of that of **A**, but both

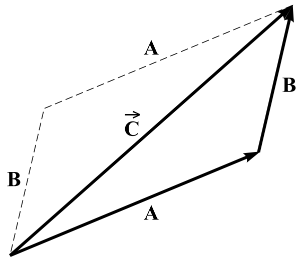


Figure 1.5. Addition of two vectors.

have the same length. Thus, subtraction of vector **B** from vector **A** is equivalent to adding $-\mathbf{B}$ to **A**:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

We see that vector addition has the following properties:

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (commutativity);
(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (associativity);
(c) $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$;
(d) $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.

We now turn to vector multiplication. Note that division by a vector is not defined: expressions such as $k/\mathbf{A}$ or $\mathbf{B}/\mathbf{A}$ are meaningless.

There are several ways of multiplying two vectors, each of which has a special meaning; two types are defined.

The scalar product

The scalar (dot or inner) product of two vectors **A** and **B** is a real number defined (in geometrical language) as the product of their magnitude and the cosine of the (smaller) angle between them (Figure 1.6):

$$\mathbf{A} \cdot \mathbf{B} \equiv AB \cos \theta \qquad (0 \le \theta \le \pi). \tag{1.4}$$

It is clear from the definition (1.4) that the scalar product is commutative:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}, \tag{1.5}$$

and the product of a vector with itself gives the square of the dot product of the vector:

$$\mathbf{A} \cdot \mathbf{A} = A^2. \tag{1.6}$$

If $\mathbf{A} \cdot \mathbf{B} = 0$ and neither **A** nor **B** is a null (zero) vector, then **A** is perpendicular to **B**.

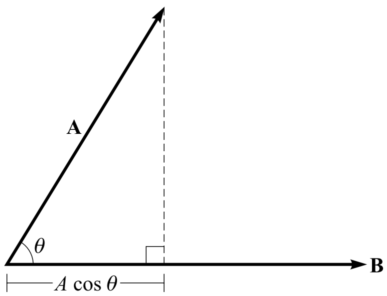


Figure 1.6. The scalar product of two vectors.

We can get a simple geometric interpretation of the dot product from an inspection of Fig. 1.6:

$$(B\cos\theta)A = \text{projection of } \mathbf{B} \text{ onto } \mathbf{A} \text{ multiplied by the magnitude of } \mathbf{A},$$

$$(A\cos\theta)B = \text{projection of } \mathbf{A} \text{ onto } \mathbf{B} \text{ multiplied by the magnitude of } \mathbf{B}.$$

If only the components of **A** and **B** are known, then it would not be practical to calculate $\mathbf{A}\cdot\mathbf{B}$ from definition (1.4). But, in this case, we can calculate $\mathbf{A}\cdot\mathbf{B}$ in terms of the components:

$$\mathbf{A}\cdot\mathbf{B} = (A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3)\cdot(B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3); \tag{1.7}$$

the right hand side has nine terms, all involving the product $\hat{e}_i \cdot \hat{e}_j$. Fortunately, the angle between each pair of unit vectors is 90°, and from (1.4) and (1.6) we find that

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}, \qquad i, j = 1, 2, 3, \tag{1.8}$$

where δ_{ij} is the Kronecker delta symbol

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases} \tag{1.9}$$

After we use (1.8) to simplify the resulting nine terms on the right-side of (7), we obtain

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 = \sum_{i=1}^{3} A_iB_i. \tag{1.10}$$

The law of cosines for plane triangles can be easily proved with the application of the scalar product: refer to Fig. 1.7, where **C** is the resultant vector of **A** and **B**. Taking the dot product of **C** with itself, we obtain

$$\begin{aligned} C^2 &= \mathbf{C}\cdot\mathbf{C} = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) \\ &= A^2 + B^2 + 2\mathbf{A}\cdot\mathbf{B} = A^2 + B^2 + 2AB\cos\theta, \end{aligned}$$

which is the law of cosines.

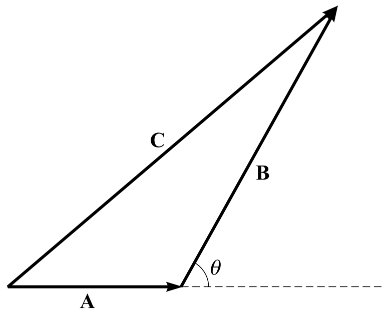


Figure 1.7. Law of cosines.

A simple application of the scalar product in physics is the work W done by a constant force $\mathbf{F}$: $W = \mathbf{F} \cdot \mathbf{r}$, where $\mathbf{r}$ is the displacement vector of the object moved by $\mathbf{F}$.

The vector (cross or outer) product

The vector product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is a vector and is written as

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}. \tag{1.11}$$

As shown in Fig. 1.8, the two vectors $\mathbf{A}$ and $\mathbf{B}$ form two sides of a parallelogram. We define $\mathbf{C}$ to be perpendicular to the plane of this parallelogram with its magnitude equal to the area of the parallelogram. And we choose the direction of $\mathbf{C}$ along the thumb of the right hand when the fingers rotate from $\mathbf{A}$ to $\mathbf{B}$ (angle of rotation less than 180°).

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = AB \sin\theta \hat{e}_C \qquad (0 \le \theta \le \pi). \tag{1.12}$$

From the definition of the vector product and following the right hand rule, we can see immediately that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}. \tag{1.13}$$

Hence the vector product is not commutative. If $\mathbf{A}$ and $\mathbf{B}$ are parallel, then it follows from Eq. (1.12) that

$$\mathbf{A} \times \mathbf{B} = 0. \tag{1.14}$$

In particular

$$\mathbf{A} \times \mathbf{A} = 0. \tag{1.14a}$$

In vector components, we have

$$\mathbf{A} \times \mathbf{B} = (A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3) \times (B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3). \tag{1.15}$$

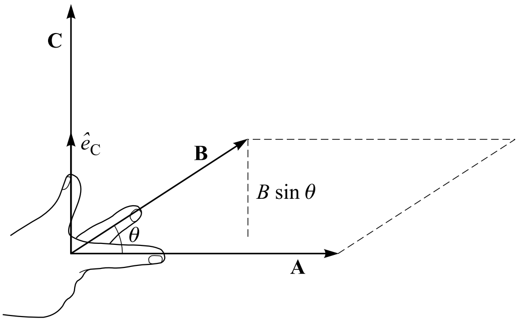


Figure 1.8. The right hand rule for vector product.

Using the following relations

$$\begin{aligned} &\hat{e}_i \times \hat{e}_i = 0, \;\; i = 1, 2, 3, \\ &\hat{e}_1 \times \hat{e}_2 = \hat{e}_3, \;\; \hat{e}_2 \times \hat{e}_3 = \hat{e}_1, \;\; \hat{e}_3 \times \hat{e}_1 = \hat{e}_2, \end{aligned} \tag{1.16}$$

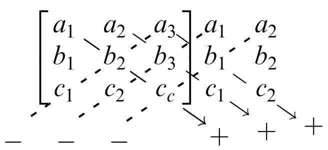
Eq. (1.15) becomes

$$\mathbf{A} \times \mathbf{B} = (A_2B_3 - A_3B_2)\hat{e}_1 + (A_3B_1 - A_1B_3)\hat{e}_2 + (A_1B_2 - A_2B_1)\hat{e}_3. \tag{1.15a}$$

This can be written as an easily remembered determinant of third order:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}. \tag{1.17}$$

The expansion of a determinant of third order can be obtained by diagonal multiplication by repeating on the right the first two columns of the determinant and adding the signed products of the elements on the various diagonals in the resulting array:



The non-commutativity of the vector product of two vectors now appears as a consequence of the fact that interchanging two rows of a determinant changes its sign, and the vanishing of the vector product of two vectors in the same direction appears as a consequence of the fact that a determinant vanishes if one of its rows is a multiple of another.

The determinant is a basic tool used in physics and engineering. The reader is assumed to be familiar with this subject. Those who are in need of review should read Appendix II.

The vector resulting from the vector product of two vectors is called an axial vector, while ordinary vectors are sometimes called polar vectors. Thus, in Eq. (1.11), $\mathbf{C}$ is a pseudovector, while $\mathbf{A}$ and $\mathbf{B}$ are axial vectors. On an inversion of coordinates, polar vectors change sign but an axial vector does not change sign.

A simple application of the vector product in physics is the torque $\boldsymbol{\tau}$ of a force $\mathbf{F}$ about a point O: $\boldsymbol{\tau} = \mathbf{F} \times \mathbf{r}$, where $\mathbf{r}$ is the vector from O to the initial point of the force $\mathbf{F}$ (Fig. 1.9).

We can write the nine equations implied by Eq. (1.16) in terms of permutation symbols ε_{ijk}:

$$\hat{e}_i \times \hat{e}_j = \varepsilon_{ijk}\hat{e}_k, \tag{1.16a}$$

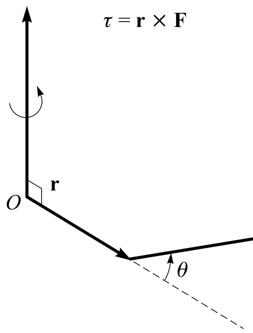


Figure 1.9. The torque of a force about a point O.

where ε_{ijk} is defined by

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3), \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3), \\ 0 & \text{otherwise (for example, if 2 or more indices are equal).} \end{cases} \tag{1.18}$$

It follows immediately that

$$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki} = -\varepsilon_{jik} = -\varepsilon_{kji} = -\varepsilon_{ikj}.$$

There is a very useful identity relating the ε_{ijk} and the Kronecker delta symbol:

$$\sum_{k=1}^{3} \varepsilon_{mnk}\varepsilon_{ijk} = \delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni}, \tag{1.19}$$

$$\sum_{j,k} \varepsilon_{mjk}\varepsilon_{njk} = 2\delta_{mn}, \quad \sum_{i,j,k} \varepsilon_{ijk}^2 = 6. \tag{1.19a}$$

Using permutation symbols, we can now write the vector product $\mathbf{A} \times \mathbf{B}$ as

$$\mathbf{A} \times \mathbf{B} = \left(\sum_{i=1}^{3} A_i \hat{e}_i\right) \times \left(\sum_{j=1}^{3} B_j \hat{e}_j\right) = \sum_{i,j}^{3} A_i B_j (\hat{e}_i \times \hat{e}_j) = \sum_{i,j,k}^{3} (A_i B_j \varepsilon_{ijk}) \hat{e}_k.$$

Thus the kth component of $\mathbf{A} \times \mathbf{B}$ is

$$(\mathbf{A} \times \mathbf{B})_k = \sum_{i,j} A_i B_j \varepsilon_{ijk} = \sum_{i,j} \varepsilon_{kij} A_i B_j.$$

If $k = 1$, we obtain the usual geometrical result:

$$(\mathbf{A} \times \mathbf{B})_1 = \sum_{i,j} \varepsilon_{1ij} A_i B_j = \varepsilon_{123} A_2 B_3 + \varepsilon_{132} A_3 B_2 = A_2 B_3 - A_3 B_2.$$

The triple scalar product A · (B × C)

We now briefly discuss the scalar $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$. This scalar represents the volume of the parallelepiped formed by the coterminous sides **A**, **B**, **C**, since

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = ABC\sin\theta\cos\alpha = hS = \text{volume},$$

S being the area of the parallelogram with sides **B** and **C**, and h the height of the parallelogram (Fig. 1.10).

Now

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3)\cdot\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1)$$

so that

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}. \tag{1.20}$$

The exchange of two rows (or two columns) changes the sign of the determinant but does not change its absolute value. Using this property, we find

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\begin{vmatrix} C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}),$$

that is, the dot and the cross may be interchanged in the triple scalar product.

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} \tag{1.21}$$

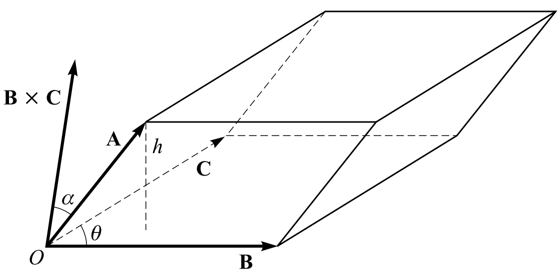


Figure 1.10. The triple scalar product of three vectors **A**, **B**, **C**.