CAMBRIDGE TRACTS IN MATHEMATICS

General Editors

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121 Lévy Processes

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011–4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock house, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 1996
First paperback edition 1998
Third printing 2004

Printed in the United Kingdom at the University Press, Cambridge

Typeset in Monotype Times 10/13pt, in TEX [TAG]

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication data applied for
Bertoin, Jean.
Lévy processes / Jean Bertoin.
p. cm. – (Cambridge tracts in mathematics ; 121)
Includes bibliographical references.
ISBN 0 521 56243 0 (hardcover)
1. Lévy processes. I. Title. II. Series.
QA274.73.B48 1996
519.2′82 – dc20 95-48226 CIP

ISBN 0 521 56243 0 hardback
ISBN 0 521 64632 4 paperback

TAG

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Preface

Lévy processes can be thought of as random walks in continuous time, that is they are stochastic processes with independent and stationary increments. The state space may be a fairly general topological group, but in this text, we will stick to the Euclidean framework. The best known and most important examples are the Poisson process, Brownian motion, the Cauchy process, and more generally stable processes. Lévy processes concern many aspects of probability theory and its applications. In particular, they are prototypes of Markov processes (actually, they form the class of space-time homogeneous Markov processes) and of semimartingales; they are also used as models in the study of queues, insurance risks, dams, and more recently in mathematical finance. From the viewpoint of functional analysis, they appear in connection with potential theory of convolution semigroups.

Historically, the first researches go back to the late 20's (that is when the foundations of modern probability theory were laid down) with the study of infinitely divisible distributions. Their general structure has been gradually discovered by de Finetti, Kolmogorov, Lévy, Khintchine and Itô; it is described by the celebrated Lévy-Khintchine formula which points out the correspondence between infinitely divisible distributions and processes with independent and stationary increments. After the pioneer contribution of Hunt in the mid-50's, the spreading of the theory of Markov processes and its connection with abstract potential theory has had a considerable impact on Lévy processes; see the works of Doob, Dynkin, Blumenthal and Getoor, Skorohod, Kesten, Bretagnolle, Port and Stone, Berg and Forst, Kanda, Hawkes, $\cdots$. At the same time, the fluctuation theory for random walks developed chiefly by Spitzer, Feller and Borovkov via analytic methods has been extended to continuous time by approximations based on discrete time skeletons; and many important properties of the sample paths of Lévy processes have been noted by Rogozin, Taylor, Fristedt, Pruitt, and others. Path transformations such as reflexion, splitting or time reversal form another set of useful techniques that were applied initially (in continuous time) to Brownian motion. Their importance for Lévy processes was recognized first by Millar, Greenwood and Pitman, who presented a direct approach to fluctuation theory. Further developments in this setting were made quite recently by Bertoin, Doney and others. Local times have received a lot of attention in the last ten years or so; the most impressive result in that field is perhaps the characterization by Barlow and Hawkes of the class of Lévy processes which possess jointly continuous local times; see also the recent works of Marcus and Rosen in the symmetric case. To complete this brief overview, we stress that the so-called general theory of processes also has many important applications to Lévy processes, in particular concerning stochastic calculus and limit theorems.

Several books contain sections or chapters on Lévy processes (e.g. Lévy (1954), Itô (1961), Gihman and Skorohod (1975), Jacod and Shiryaev (1987), Sato (1990, 1995), Skorohod (1991), Rogers and Williams (1994), …); see also the surveys by Taylor (1973), Fristedt (1974) and Bingham (1975). The purpose of this monograph is to present an up-dated and concise account of the theory, which may serve as a reference text. I endeavoured to make it as self-contained as possible; the prerequisite is limited to standard notions in probability and Fourier analysis.

Here is a short description of the content. A chapter of preliminaries introduces the notation and reviews some elementary material on infinitely divisible laws, Poisson processes, martingales, Brownian motion and regularly varying functions. The core of the theory of Lévy processes in connection with the Markov property and the related potential theory is developed in chapters I and II. The theory of general Markov processes is doubtless one of the most fascinating fields of probability, but it is also one of the most demanding. Nonetheless, the special case of Lévy processes is much easier to handle, thanks to techniques of Fourier analysis and the spatial homogeneity. We stress that no prior knowledge of Markov processes is assumed. Chapter III is devoted to the study of subordinators, which form the class of increasing Lévy processes; a special emphasis is given to the properties of their sample paths. Subordinators also have a key part in chapter IV, where we introduce Itô's theory of the excursions of a Markov process away from a point, and in chapter V, where we investigate the local times of Lévy processes. The fluctuation theory is presented in chapter VI, following the Greenwood-Pitman approach based on excursion theory. Chapter VII is devoted to Lévy processes with no positive jumps, for which fluctuation theory becomes remarkably simple. Some path transformations are described, which extend well-known identities for Brownian motion due to Williams and Pitman. Finally, several consequences of the scaling property of stable processes are presented in chapter VIII. Each chapter ends with exercises, which provide additional information on the topic for the interested reader, and with comments, where credits and further references are given. To avoid duplication with the existing literature on semimartingales, we did not include material on stochastic calculus or limit theorems; we refer to Jacod (1979), Protter (1990) and Jacod and Shiryaev (1987) for detailed expositions.

We use the following labels. Roman numbers refer to chapters, arabic numbers to statements, and numbers between parentheses to equations or formulas. For instance, Proposition V.2 designates the proposition with label 2 of chapter V, and (III.10) the equation with label (10) in chapter III. The roman number refering to a chapter is omitted within the same chapter.

This text is partially based on a 'cours de troisième cycle' taught in the Laboratoire de Probabilités de L'Université Pierre-et-Marie-Curie. My work was greatly eased by the position I had at this time in the Centre National de la Recherche Scientifique. I should like to thank warmly my colleagues in the Laboratoire de Probabilités, and to express my deep gratitude to Nick Bingham, Ron Doney, Daniel Revuz and Hrvoje Sikic, who read preliminary versions of the manuscript and corrected uncountable errors, misprints and misuses of the English language.