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Preliminaries

1. Notation

In this section, we set down notation which will be used throughout the text.

We denote by $\mathbb{R}^d$ the d-dimensional Euclidean space, equipped with the standard scalar product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $|\cdot|$. It is endowed with the Borel sigma-field $\mathscr{B}(\mathbb{R}^d)$ and the Lebesgue measure dx. The abbreviation a.e. refers to 'almost everywhere' with respect to the Lebesgue measure. The lower and upper bounds of a subset A of the nonnegative half-line $[0, \infty)$ are denoted by $\inf A$ and $\sup A$, respectively, with the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = 0$. We say that a function $f : [0, \infty) \to [0, \infty]$ is increasing if $f(s) \leq f(t)$ for all $0 \leq s \leq t$. If the preceding condition holds with $\leq$ replaced by $<$, we say that f is strictly increasing. We use Landau's notation $f = o(g)$, $f = O(g)$ and $f \sim g$ for $\lim(f/g) = 0$, $\limsup(f/g) < \infty$ and $\lim(f/g) = 1$, respectively.

Next, we introduce the so-called canonical notation for right-continuous substochastic (i.e. possibly defective) processes having left limits. Specifically, take an isolated point ∂ which will serve as cemetery. Consider

$$\Omega = D([0, \infty), \mathbb{R}^d \cup \{\partial\}),$$

the set of paths $\omega : [0, \infty) \to \mathbb{R}^d \cup \{\partial\}$ with lifetime

$$\zeta(\omega) = \inf\{t \geq 0 : \omega(t) = \partial\}$$

which are right-continuous on $[0, \infty)$, have a left limit denoted by $\omega(s-)$ for any $s \in (0, \infty)$, and stay at the cemetery point ∂ after the lifetime $\zeta(\omega)$. This space is endowed with Skorohod's topology, for which we refer to chapter VI in Jacod and Shiryaev (1987). In particular, Ω is a Polish space, that is it is metric-complete and separable. We shall not use Skorohod's topology directly, but it is crucial to work on a Polish space to apply fundamental theorems of probability theory, such as the existence of conditional laws. The Borel sigma-field of Ω is denoted by $\mathscr{F}$.

We then introduce the coordinate process $X = (X_t, t \geq 0)$, where

$$X_t = X_t(\omega) = \omega(t).$$

We also write $\zeta = \zeta(\omega)$ for the lifetime of X and

$$X_{s-} = X_{s-}(\omega) = \omega(s-) \quad , \quad \Delta X_s = X_s - X_{s-}$$

respectively for the left limit and the jump at time $s \in (0, \zeta)$. The family of mappings $\theta_t : \Omega \to \Omega$ and $\mathrm{k}_t : \Omega \to \Omega$ $(t \geq 0)$, specified by

$$\theta_t \omega(s) = \omega(t+s) \qquad (s \geq 0)$$

and

$$\mathrm{k}_t \omega(s) = \begin{cases} \omega(s) & \text{if } s < t\,, \\ \partial & \text{otherwise} \end{cases}$$

are called the translation and the killing operators, respectively.

Suppose that $\mathbb{P}$ is a probability measure on $(\Omega, \mathscr{F})$, and Y is a random variable, say taking values in $\mathbb{R}^d$. We denote the expectation of Y under $\mathbb{P}$ by $\mathbb{E}(Y)$ whenever it makes sense. We then write $\mathbb{E}(Y, \Lambda_1, \cdots, \Lambda_k)$ for $\mathbb{E}(\mathbf{1}_\Lambda Y)$ with $\Lambda = \Lambda_1 \cap \cdots \cap \Lambda_k$, where $\Lambda_1, \cdots, \Lambda_k \in \mathscr{F}$, and $\mathbb{E}(Y \mid \mathscr{G})$ for the conditional expectation given some subfield $\mathscr{G}$. Finally, we denote either by $\mathbb{P}(Y \in \cdot)$ or by $\mathbb{P}(Y \in dy)$ the distribution of Y under $\mathbb{P}$. We say that a family $(\mathbb{P}(\cdot | Y = y), y \in \mathbb{R}^d)$ of laws on $(\Omega, \mathscr{F})$ is a version of the conditional law $\mathbb{P}$ given Y if the mapping $y \to \mathbb{P}(\cdot | Y = y)$ is measurable, $\mathbb{P}(Y = y | Y = y) = 1$ for all $y \in \mathbb{R}^d$, and

$$\mathbb{P}(\Lambda) = \int_{\mathbb{R}^d} \mathbb{P}(\Lambda \mid Y = y) \mathbb{P}(Y \in dy)\,, \qquad \Lambda \in \mathscr{F}.$$

We refer e.g. to chapter III in Dellacherie and Meyer (1975) for the existence of conditional laws.

2. Infinitely divisible distributions

Consider a probability measure μ on $\mathbb{R}^d$, and its characteristic function

$$\mathscr{F}\mu(\lambda) = \int_{\mathbb{R}^d} \exp\{\mathrm{i}\langle \lambda, x \rangle\} \mu(dx) \quad (\lambda \in \mathbb{R}^d).$$

The law μ is called *infinitely divisible* if for any positive integer n, there exists a probability measure μ_n with characteristic function $\mathscr{F}\mu_n$ such that $\mathscr{F}\mu = (\mathscr{F}\mu_n)^n$. In other words, μ can be expressed as the n-th convolution power of μ_n. The simplest examples of infinitely divisible laws are Dirac point masses, Gaussian and stable distributions, and in dimension $d = 1$, Poisson and Gamma distributions.

Assume now that μ is infinitely divisible. Then its characteristic function never vanishes and can be expressed as follows. There is a unique continuous function $\Psi : \mathbb{R}^d \to \mathbb{C}$, called the *characteristic exponent* of μ, such that $\Psi(0) = 0$ and

$$\mathscr{F}\mu(\lambda) = \exp\{-\Psi(\lambda)\} \quad (\lambda \in \mathbb{R}^d).$$

We see that if μ_1 and μ_2 are two infinitely divisible laws with respective characteristic exponents Ψ_1 and Ψ_2, then the convolution $\mu_1 \star \mu_2$ is again infinitely divisible with characteristic exponent $\Psi_1 + \Psi_2$.

The starting point of many studies of infinitely divisible laws is the famous Lévy-Khintchine formula (see for instance section 7.6 in Chung (1968), or chapter XVII in Feller (1971)) which determines the class of characteristic functions corresponding to infinitely divisible laws.

Lévy-Khintchine formula *A function $\Psi : \mathbb{R}^d \to \mathbb{C}$ is the characteristic exponent of an infinitely divisible probability measure on $\mathbb{R}^d$ if and only if there are $a \in \mathbb{R}^d$, a positive semi-definite quadratic form Q on $\mathbb{R}^d$, and a measure Π on $\mathbb{R}^d - \{0\}$ with $\int(1 \wedge |x|^2)\Pi(dx) < \infty$ such that*

$$\Psi(\lambda) = \mathrm{i}\langle a, \lambda\rangle + \frac{1}{2}Q(\lambda) + \int_{\mathbb{R}^d} \left(1 - \mathrm{e}^{\mathrm{i}\langle\lambda,x\rangle} + \mathrm{i}\langle\lambda, x\rangle \mathbf{1}_{\{|x|<1\}}\right)\Pi(dx) \qquad (1)$$

for every $\lambda \in \mathbb{R}^d$.

The parameters a, Q, and Π appearing in (1) are determined by Ψ, and their probabilistic meanings will be clarified in section I.1. The measure Π is called the *Lévy measure* of μ and the quadratic form Q the *Gaussian coefficient* . We mention that some authors use a slightly different expression for the Lévy-Khintchine formula. Specifically, the cut-off function $\mathbf{1}_{\{|x|<1\}}$ is replaced by a bounded smooth function which is equivalent to 1 at the origin, the most common being $(1 + |x|^2)^{-1}$. Such a change in the choice of the cut-off function does not alter the Lévy measure and the Gaussian coefficient, but the parameter a has to be replaced by

$$a' = a + \int_{\mathbb{R}^d} x \left(\frac{1}{1 + |x|^2} - \mathbf{1}_{\{|x|<1\}}\right) \Pi(dx).$$

3. Martingales

Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with a filtration $(\mathscr{F}_t)_{t\geq 0}$, i.e. an increasing family of sub-fields, which fulfils the usual conditions. That is each $\mathscr{F}_t$ is $\mathbb{P}$-complete and $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$ for every t. A real-valued stochastic process $M = (M_t, 0 \leq t < \infty)$ is a *martingale* if

$$\mathbb{E}(M_t \mid \mathscr{F}_s) = M_s, \qquad 0 \leq s \leq t.$$

(It is implicit here that $\mathbb{E}(|M_t|) < \infty$ for all t.) We say that M is right-continuous if its sample paths are right-continuous a.s., and *uniformly integrable* if there exists an increasing function $f : [0, \infty) \to [0, \infty)$ with $x = o(f(x))$ as x goes to ∞, such that $\sup\{\mathbb{E}(f(|M_t|)) : t \geq 0\} < \infty$.

We assume from now on that M is a right-continuous martingale. The following key results are due to Doob, and we refer to chapter VI of Dellacherie and Meyer (1980) for a complete account.

Maximal inequality *For every $t > 0$, we have*

$$\mathbb{E}\left(\sup\{|M_s|^2 : 0 \leq s \leq t\}\right) \leq 4\mathbb{E}(|M_t|^2).$$

A nonnegative random variable T is called a *stopping time* if for every $t \geq 0$, $\{T \leq t\} \in \mathcal{F}_t$.

Optional sampling theorem *Suppose that T is a stopping time, a.s. finite.*

(i) *The stopped process $(M_{T\wedge t}, t \geq 0)$ is again a martingale.*
(ii) *Suppose moreover that M is uniformly integrable. Then $\mathbb{E}(M_T) = \mathbb{E}(M_0)$.*

Convergence theorem *Suppose that M is uniformly integrable. Then $\lim_{t\to\infty} M_t = M_\infty$ exists a.s. and in $L^1(\mathbb{P})$, and $M_t = \mathbb{E}(M_\infty \mid \mathcal{F}_t)$ for all t.*

4. Poisson processes

The proofs of the results stated in this section and the next can be found in section XII.1 in Revuz and Yor (1994).

The Poisson distribution with parameter (or intensity) $c > 0$ is the probability measure on integers which assigns mass $e^{-c}c^k/(k!)$ at point $k \in \mathbb{N}$. Its characteristic function is

$$\sum_{k=0}^{\infty} e^{i\lambda k} e^{-c}\frac{c^k}{k!} = \exp\{-c(1-e^{i\lambda})\} \quad , \quad \lambda \in \mathbb{R}.$$

The Poisson distribution is infinitely divisible, and the results of section I.1 below guarantee the existence of a unique (in law) increasing right-continuous process N with stationary independent increments, called a Poisson process of parameter (or intensity) c, such that for each $t > 0$, N_t has a Poisson distribution with parameter ct. One can also construct N directly as follows. Consider a probability measure $\mathbb{P}$ and a sequence $\tau_1, \cdots, \tau_n, \cdots$ of independent exponential variables with parameter c, that is $\mathbb{P}(\tau_i > s) = e^{-cs}$ for $s \geq 0$. Introduce the partial sums $S_n = \tau_1 + \cdots + \tau_n$, $n \in \mathbb{N}$, so that S_n has the Gamma(c, n) distribution,

$$\mathbb{P}(S_n \in ds) = \frac{c^n}{(n-1)!} s^{n-1} e^{-cs}\, ds \qquad (s \geq 0)\,.$$

Then consider the right-continuous inverse $N_t = \sup\{n \in \mathbb{N} : S_n \leq t\}$ $(t \geq 0)$, so that for every $t \geq 0$ and $k \in \mathbb{N}$,

$$\mathbb{P}(N_t = k) = \mathbb{P}(S_k \leq t, S_{k+1} > t) = \int_0^t \frac{c^k}{(k-1)!} s^{k-1} \mathrm{e}^{-cs} \mathrm{e}^{-c(t-s)}\, ds$$
$$= \mathrm{e}^{-ct}(ct)^k/(k!)\,.$$

On the other hand, it follows easily from the so-called lack-of-memory property of the exponential law that for every $0 \leq s \leq t$, the increment $N_{t+s} - N_t$ has the Poisson distribution with parameter cs and is independent of the sigma-field generated by $(N_u, u \leq t)$.

Next, let $(\mathscr{G}_t)$ be a filtration which satisfies the usual conditions. We say that N is a $(\mathscr{G}_t)$-Poisson process if N is a Poisson process which is adapted to $(\mathscr{G}_t)$ and for every $s, t \geq 0$, the increment $N_{t+s} - N_t$ is independent of $\mathscr{G}_t$. In particular, N is a $(\mathscr{G}_t)$-Poisson process if $(\mathscr{G}_t)$ is the natural filtration of N.

There are three important families of martingales related to a $(\mathscr{G}_t)$-Poisson process. First, one says that a process $H = (H_t, t \geq 0)$ is *predictable* if it is measurable in the sigma-field generated by the left-continuous adapted processes. If H is a real-valued predictable process with $\mathbb{E}(\int_0^t |H_s| ds) < \infty$ for all $t \geq 0$ and if $N = (N_t, t \geq 0)$ is a $(\mathscr{G}_t)$-Poisson process with parameter $c > 0$, then the *compensated integral*

$$M_t = \int_0^t H_s dN_s \; - \; c \int_0^t H_s ds \quad (t \geq 0)$$

is a $(\mathscr{G}_t)$-martingale. If moreover $\mathbb{E}(\int_0^t H_s^2 ds) < \infty$, then

$$M_t^2 - c\int_0^t H_s^2 ds \quad (t \geq 0)$$

is also a martingale. Finally, if H is predictable and bounded, then the same holds for the exponential process

$$\exp\left\{\int_0^t H_s dN_s + c\int_0^t (1 - \mathrm{e}^{H_s}) ds\right\} \quad (t \geq 0)\,.$$

Here, the various integrals with dN_s as integrator are taken in the sense of Stieltjes.

We conclude this section by recalling a well-known criterion for the independence of Poisson processes.

Proposition 1 *Let $N^{(i)}, i = 1, \cdots, d$, be $(\mathscr{G}_t)$-Poisson processes. They are independent if and only if they never jump simultaneously, that is for* *every i, j with $i \neq j$*

$$N_t^{(i)} - N_{t-}^{(i)} = 0 \text{ or } N_t^{(j)} - N_{t-}^{(j)} = 0 \quad \text{for all } t > 0, \text{ a.s.},$$

where $N_{t-}^{(k)}$ stands for the left limit of $N^{(k)}$ at time t.

It is crucial in Proposition 1 to assume that the $N^{(i)}$ are Poisson processes in the same filtration. Otherwise, it is easy to construct Poisson processes which never jump simultaneously and which are not independent.

5. Poisson measures and Poisson point processes

Let E be a Polish space and ν a sigma-finite measure on E. We call a random measure φ on E *a Poisson measure with intensity* ν if it satisfies the following. For every Borel subset B of E with $\nu(B) < \infty$, $\varphi(B)$ has a Poisson distribution with parameter $\nu(B)$, and if $B_1, \cdots, B_n$ are disjoint Borel sets, the variables $\varphi(B_1), \cdots, \varphi(B_n)$ are independent. Plainly, φ is then a sum of Dirac point masses.

One can construct Poisson measures as follows. First, assume that the total mass of ν is finite, and put $c = \nu(E)$. Let $\xi_1, \cdots, \xi_n, \cdots$ be a sequence of independent identically distributed random variables with common law $c^{-1}\nu$ and a Poisson variable N with parameter c independent of the ξ_n's. The random measure

$$\varphi = \sum_{j=1}^{N} \delta_{\xi_j},$$

where δ_ϵ stands for the Dirac point mass at $\epsilon \in E$, is a Poisson measure with intensity ν. If ν is merely sigma-finite, there exists a partition $(E_n, n \in \mathbb{N})$ of E into Borel sets such that $\nu(E_n) < \infty$ for every integer n. Then we can construct a sequence φ_n of independent Poisson measures with respective characteristic measures $\mathbf{1}_{E_n}\nu$, and $\varphi = \sum_n \varphi_n$ is a Poisson measure with intensity ν.

We then consider the product space $E \times [0, \infty)$, the measure $\mu = \nu \otimes dx$, and a Poisson measure φ on $E \times [0, \infty)$ with intensity μ. It is easy to check that a.s., $\varphi(E \times \{t\}) = 0$ or 1 for all $t \geq 0$. This enables us to represent φ in terms of a stochastic process taking values in $E \cup \{\Upsilon\}$, where Υ is an isolated additional point. Specifically, if $\varphi(E \times \{t\}) = 0$, then put $e(t) = \Upsilon$. If $\varphi(E \times \{t\}) = 1$, then the restriction of φ to the section $E \times \{t\}$ is a Dirac point mass, say at (ϵ, t), and we put $e(t) = \epsilon$. We can now express the Poisson measure as

$$\varphi = \sum_{t \geq 0} \delta_{(e(t), t)}.$$

The process $e = (e(t), t \geq 0)$ is called a *Poisson point process* with characteristic measure ν. We denote its natural filtration after the usual completion by $(\mathcal{G}_t)$.

For every Borel subset B of E, we call

$$N_t^B = \text{Card}\{s \leq t : e(s) \in B\} = \varphi(B \times [0, t]) \qquad (t \geq 0)$$

the *counting process* of B. It is a $(\mathcal{G}_t)$-Poisson process with parameter $\nu(B)$. Conversely, suppose that $e = (e(t), t \geq 0)$ is a stochastic process taking values in $E \cup \{\Upsilon\}$ such that, for every Borel subset B of E, the counting process $N_t^B = \text{Card}\{s \leq t : e(s) \in B\}$ is a Poisson process with intensity $\nu(B)$ in a given filtration $(\mathcal{G}_t)$. Then observe that counting processes associated to disjoint Borel sets never jump simultaneously and thus are independent according to Proposition 1. One then deduces that the associated random measure $\varphi = \sum_{t \geq 0} \delta_{(e(t),t)}$ is a Poisson measure with intensity μ.

We next present a useful probabilistic interpretation of the characteristic measure ν.

Proposition 2 *Let B be a Borel set with $0 < \nu(B) < \infty$. The first entrance time of e into B, $T_B = \inf\{t \geq 0 : e(t) \in B\}$, is a $(\mathcal{G}_t)$-stopping time and we have*

(i) *T_B has an exponential distribution with parameter $\nu(B)$.*

(ii) *The random variable $e(T_B)$ is independent of T_B and has the law $\nu(\cdot \mid B)$, that is for every Borel set A,*

$$\mathbb{P}(e(T_B) \in A) = \nu(A \cap B)/\nu(B)\,.$$

(iii) *The process e' given by $e'(t) = \Upsilon$ if $e(t) \in B$ and $e'(t) = e(t)$ otherwise $(t \geq 0)$ is a Poisson point process with characteristic measure $1_{B^c}\nu$, and is independent of $(T_B, e(T_B))$.*

The process $(e_t, 0 \leq t \leq T_B)$ is called stopped at the first point in B, its law is characterized by Proposition 2.

In practice, it is important to calculate certain expressions in terms of the characteristic measure. The following two formulas are the most useful:

Compensation formula *Let $H = (H_t, t \geq 0)$ be a predictable process taking values in the space of nonnegative measurable functions on $E \cup \{\Upsilon\}$, such that $H_t(\Upsilon) = 0$ for all $t \geq 0$. We have*

$$\mathbb{E}\left(\sum_{0 \leq t < \infty} H_t(e(t))\right) = \mathbb{E}\left(\int_0^\infty dt \int_E d\nu(\epsilon) H_t(\epsilon)\right).$$

Exponential formula *Let f be a complex-valued Borel function on $E \cup \{\Upsilon\}$ with $f(\Upsilon) = 0$ and*

$$\int_E \nu(d\epsilon)|1 - e^{f(\epsilon)}| < \infty .$$

We have for every $t \geq 0$

$$\mathbb{E}\left(\exp\left\{\sum_{0 \leq s \leq t} f(e(s))\right\}\right) = \exp\left\{-t \int_E \nu(d\epsilon)(1 - e^{f(\epsilon)})\right\} .$$

These two formulas are easy to prove when the space E is finite, using respectively the first and the third special martingale of section 4. The general case then follows from a monotone class theorem.

We conclude this section with a useful inequality which is a consequence of Doob's maximal inequality applied to the first special martingale of section 4.

Maximal inequality for compensated sums *Let f be a Borel function on $E \cup \{\Upsilon\}$ with $f(\Upsilon) = 0$. We have for every fixed $T > 0$*

$$\mathbb{E}\left(\sup\left\{\left|\sum_{0 \leq s \leq t} f(e(s)) - t\int_E f(\epsilon)d\nu(\epsilon)\right|^2, 0 \leq t \leq T\right\}\right) \leq 4T \int_E f(\epsilon)^2 d\nu(\epsilon) .$$

6. Brownian motion

A real-valued stochastic process $B = (B_t, t \geq 0)$ is a (linear) *Brownian motion* if its sample paths are continuous a.s., its law at any fixed time $t > 0$ is the centred Gaussian distribution with variance t,

$$\mathbb{P}(B_t \in dx) = (2\pi t)^{-1/2} \exp\{-x^2/2t\} dx ,$$

and its increments are independent in the sense that for any $s, t > 0$, $B_{t+s} - B_t$ is independent of the σ-field generated by $(B_u, 0 \leq u \leq t)$. Note that this implies that $B_{t+s} - B_t$ has the centred Gaussian distribution with variance s, so that B is a Gaussian process with stationary (or, homogeneous) independent increments. Finally, a process $(B^1, \cdots, B^d)$ taking values in the d-dimensional Euclidean space is a Brownian motion if its coordinates $B^1, \cdots, B^d$ are independent linear Brownian motions.

There are several different constructions of Brownian motion; here is one of the simplest (see e.g. section I.1 in Revuz and Yor (1994)). First, a standard result guarantees the existence of a centred Gaussian process $\widetilde{B} = (\widetilde{B}_t, t \geq 0)$ with covariance $\mathbb{E}(\widetilde{B}_t \widetilde{B}_s) = s \wedge t$. Then one applies Kolmogorov's criterion to verify that there is a continuous version B of $\widetilde{B}$, that is $B_t = \widetilde{B}_t$ a.s. for every $t \geq 0$. Actually, Kolmogorov's criterion shows that the sample paths of B are a.s. Hölder-continuous with order $1/2-\varepsilon$ on every compact time interval and for every $\varepsilon > 0$. On the other hand, B is a.s. nowhere Hölder-continuous of order $1/2+\varepsilon$. In particular, B is nowhere differentiable and its total variation is infinite a.s. on any non-trivial time interval.

7. Regular variation and Tauberian theorems

In this section, we recall the basis of Karamata's theory, and refer to the first chapter of Bingham, Goldie and Teugels (1987) for details and proofs.

A measurable function $\ell : (0,\infty) \to (0,\infty)$ is *slowly varying* at $0+$ (respectively, at ∞) if for every $\lambda > 0$, $\lim(\ell(\lambda x)/\ell(x)) = 1$ as x tends to $0+$ (respectively, to ∞).

Representation of a slowly varying function *The function ℓ is slowly varying at $0+$ (respectively, at ∞) if and only if it may be written in the form*

$$\ell(x) = \exp\left\{c(x) + \int_1^x \frac{\varepsilon(u)du}{u}\right\}$$

where $c, \varepsilon : (0,\infty) \to \mathbb{R}$ are two bounded measurable functions with $\lim c(x) = d \in \mathbb{R}$ and $\lim \varepsilon(x) = 0$ as $x \to 0+$ (respectively, as $x \to \infty$).

A measurable function $f : (0,\infty) \to (0,\infty)$ is *regularly varying* at $0+$ (respectively, at ∞) if for every $\lambda > 0$, the ratio $f(\lambda x)/f(x)$ converges in $(0,\infty)$ as x tends to $0+$ (respectively, to ∞).

Characterization of a regularly varying function *If the function f is regularly varying at $0+$ (respectively, at ∞), then there exists a real number ρ, called the index, such that*

$$\lim(f(\lambda x)/f(x)) = \lambda^\rho \qquad (x \to 0+\ ,\ \textit{resp.}\ \infty)$$

for every $\lambda > 0$. Moreover, $\ell(x) = f(x)x^{-\rho}$ is slowly varying at $0+$ (respectively, at ∞).

Suppose now that $U : [0,\infty) \to [0,\infty)$ is an increasing right-continuous function, denote by $U(dx)$ the associated Stieltjes measure (by convention, this measure assigns a mass $U(0)$ at the origin) and by $\mathscr{L}U$ its Laplace transform,

$$\mathscr{L}U(\lambda) = \int_{[0,\infty)} e^{-\lambda x} U(dx) \in [0,\infty] \qquad (\lambda \geq 0).$$

Tauberian theorem *Let $\ell : (0,\infty) \to (0,\infty)$ be slowly varying at $0+$ (respectively, at ∞) and $\rho \geq 0$. The following are equivalent:*

(i) $$U(x) \sim x^{\rho}\ell(x)/\Gamma(1+\rho) \qquad (x \to 0+\ , \textit{resp. } \infty);$$

(ii) $$\mathscr{L}U(\lambda) \sim \lambda^{-\rho}\ell(1/\lambda) \qquad (\lambda \to \infty, \textit{ resp. } 0+).$$

It easy to see that if U is regularly varying with index $\rho > 0$, then its inverse function is again regularly varying, but with index $1/\rho$. Another useful property concerns the case when the Stieltjes measure $U(dx)$ is absolutely continuous with a monotone density.

Monotone density theorem *Suppose that $U(dx) = u(x)dx$, where $u : (0,\infty) \to (0,\infty)$ is monotone on some neighbourhood of $0+$ (respectively, of ∞). If there exist a positive real number ρ and a function $\ell : (0,\infty) \to (0,\infty)$ that is slowly varying at $0+$ (respectively, at ∞) such that*

$$U(x) \sim x^{\rho}\ell(x) \qquad (x \to 0+\ , \textit{resp. } \infty),$$

then

$$u(x) \sim \rho x^{\rho-1}\ell(x) \qquad (x \to 0+\ , \textit{resp. } \infty).$$

Conversely, it is easy to show that if $U(dx)$ is absolutely continuous with a density that is regularly varying with index $\rho - 1$ for some $\rho > 0$, then U is regularly varying with index ρ and $U(x) \sim \rho^{-1}xu(x)$.

Regularly varying functions often appear in probability in connection with weak limit theorems. Here is a classical example which has a particular interest for us. Let $(\xi_n, n \in \mathbb{N})$ be a sequence of independent identically distributed random variables taking values in $\mathbb{R}^d$. Suppose that for some sequence $(a_n, n \in \mathbb{N})$ of positive real numbers, the renormalized sum $a_n^{-1}(\xi_1 + \cdots + \xi_n)$ converges in distribution to some non-degenerate law ν (ν is not the point mass at 0), as n goes to ∞. Then ν is a strictly stable law of index $\alpha \in (0,2]$, that is an infinitely divisible law whose characteristic exponent Ψ fulfils $\Psi(\lambda) = |\lambda|^{\alpha}\Psi\left(\lambda/|\lambda|\right)$ for all $\lambda \in \mathbb{R}^d - \{0\}$. Moreover, there exists a function $a : (0,\infty) \to (0,\infty)$ which is regularly varying at ∞ with index $1/\alpha$ and such that $a_n = a(n)$ for all $n \in \mathbb{N}$.