1

Simple Wave Solutions

Synopsis

Chapter 1 provides the background, both the model equations and some of the mathematical transformations, needed to understand linear elastic waves. Only the basic equations are summarized, without derivation. Both Fourier and Laplace transforms and their inverses are introduced and important sign conventions settled. The Poisson summation formula is also introduced and used to distinguish between a propagating wave and a vibration of a bounded body.

A linear wave carries information at a particular velocity, the group velocity, which is characteristic of the propagation structure or environment. It is this transmitting of information that gives linear waves their special importance. In order to introduce this aspect of wave propagation, propagation in one-dimensional periodic structures is discussed. Such structures are dispersive and therefore transmit information at a speed different from the wavespeed of their individual components.

1.1 Model Equations

The equations of linear elasticity consist of (1) the conservation of linear and angular momentum, and (2) a constitutive relation relating force and deformation. In the linear approximation density ρ is constant. The conservation of mechanical energy follows from (1) and (2). The most important feature of the model is that the force exerted across a surface, oriented by the unit normal n_j, by one part of a material on the other is given by the traction $t_i = \tau_{ji} n_j$, where τ_{ji} is the stress tensor. The conservation of angular momentum makes the stress tensor symmetric; that is, $\tau_{ij} = \tau_{ji}$. The conservation of linear momentum, in differential form, is expressed as

$$\partial_k \tau_{ki} + \rho f_i = \rho \partial_t \partial_t u_i . \tag{1.1}$$

The term $\boldsymbol{f}$ is a force per unit mass.

Deformation is described by using a strain tensor,

$$\epsilon_{ij} = (\partial_i u_j + \partial_j u_i)/2, \tag{1.2}$$

where u_i is ith component of particle displacement. The symmetrical definition of the deformation ensures that no rigid-body rotations are included. However, the underlying dependence of the deformation is upon the $\nabla \boldsymbol{u}$. For a homogeneous, isotropic, linearly elastic solid, stress and strain are related by

$$\tau_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}, \tag{1.3}$$

where λ and μ are Lamé's elastic constants. Substituting (1.2) in (1.3), followed by substituting the outcome into (1.1), gives one form of the equation of motion, namely

$$(\lambda + \mu)\partial_i \partial_k u_k + \mu \partial_j \partial_j u_i + \rho f_i = \rho \partial_t \partial_t u_i . \tag{1.4}$$

Written in vector notation, the equation becomes

$$(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u} + \rho \boldsymbol{f} = \rho \partial_t \partial_t \boldsymbol{u}. \tag{1.5}$$

When the identity $\nabla^2 \boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla \wedge \nabla \wedge \boldsymbol{u}$ is used, the equation can also be written in the form

$$(\lambda + 2\mu)\nabla(\nabla \cdot \boldsymbol{u}) - \mu \nabla \wedge \nabla \wedge \boldsymbol{u} + \rho \boldsymbol{f} = \rho \partial_t \partial_t \boldsymbol{u}. \tag{1.6}$$

This last equation indicates that elastic waves have both dilitational $\nabla \cdot \boldsymbol{u}$ and rotational $\nabla \wedge \boldsymbol{u}$ deformations.

If $\partial \mathcal{R}$ is the boundary of a region $\mathcal{R}$ occupied by a solid, then commonly $\boldsymbol{t}$ and $\boldsymbol{u}$ are prescribed on $\partial \mathcal{R}$. When $\boldsymbol{t}$ is given over part of $\partial \mathcal{R}$ and $\boldsymbol{u}$ over another part, the boundary conditions are said to be mixed. One very common boundary condition is to ask that $\boldsymbol{t} = 0$ everywhere on $\partial \mathcal{R}$. This models the case in which a solid surface is adjacent to a gas of such small density and compressibility that it is almost a vacuum. When $\mathcal{R}$ is infinite in one or more dimensions, special conditions are imposed such that a disturbance decays to zero at infinity or radiates outward toward infinity from any sources contained within $\mathcal{R}$.

Another common situation is that in which $\partial\mathcal{R}_{12}$ is the boundary between two regions, 1 and 2, occupied by solids having different properties. Contact between solid bodies is quite complicated, but in many cases it is usual to assume that the traction and displacement, $\boldsymbol{t}$ and $\boldsymbol{u}$, are continuous. This models welded contact. One other simple continuity condition that commonly arises is that between a solid and an ideal fluid. Because the viscosity is ignored, the tangential component of $\boldsymbol{t}$ is set to zero, while the normal component of traction and the normal component of displacement are made continuous.

These are only models and are often inadequate. To briefly indicate some of the possible complications, consider two solid bodies pressed together. A (linear) wave incident on such a boundary would experience continuity of traction and displacement when the solids press together, but would experience a traction-free boundary condition when they pull apart (Comninou and Dundurs, 1977). This produces a complex nonlinear interaction.

The reader may consult Hudson (1980) for a succinct discussion of linear elasticity or Atkin and Fox (1980) for a somewhat more general view.

1.1.1 One-Dimensional Models

We assume that the various wavefield quantities depend only on the variables x_1 and t. For *longitudinal strain*, u_1 is finite, while u_2 and u_3 are assumed to be zero, so that (1.2) combined with (1.3) becomes

$$\tau_{11} = (\lambda + 2\mu)\partial_1 u_1, \qquad \tau_{22} = \tau_{33} = \lambda \partial_1 u_1, \tag{1.7}$$

and (1.1) becomes

$$(\lambda + 2\mu)\partial_1\partial_1 u_1 + \rho f_1 = \rho \partial_t \partial_t u_1. \tag{1.8}$$

For *longitudinal stress*, all the stress components except τ_{11} are assumed to be zero. Now (1.3) becomes

$$\tau_{11} = E\partial_1 u_1, \qquad E = \mu\frac{3\lambda + 2\mu}{\lambda + \mu}, \tag{1.9}$$

and

$$\partial_2 u_2 = \partial_3 u_3 = -\nu\partial_1 u_1, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)}. \tag{1.10}$$

Now (1.1) becomes

$$E\partial_1\partial_1 u_1 + \rho f_1 = \rho\partial_t\partial_t u_1. \tag{1.11}$$

Note that (1.8) and (1.11) are essentially the same, though they have somewhat different physical meanings. The longitudinal stress model is useful for rods having a small cross section and a traction-free surface. Stress components that vanish at the surface are assumed to be negligible in the interior.

1.1.2 Two-Dimensional Models

Let us assume that the various wavefield quantities are independent of x_3. As a consequence, (1.1) breaks into two separate equations, namely

$$\partial_\beta \tau_{\beta 3} + \rho f_3 = \rho \partial_t \partial_t u_3, \tag{1.12}$$

$$\partial_\beta \tau_{\beta\alpha} + \rho f_\alpha = \rho \partial_t \partial_t u_\alpha. \tag{1.13}$$

Greek subscripts $\alpha, \beta = 1, 2$ are used to indicate that the independent spatial variables are x_1 and x_2. The case for which the only nonzero displacement component is $u_3(x_1, x_2, t)$, namely (1.12), is called *antiplane shear* motion, or SH motion for shear horizontal.

$$\tau_{3\beta} = \mu \partial_\beta u_3, \tag{1.14}$$

giving, from (1.12),

$$\mu \partial_\beta \partial_\beta u_3 + \rho f_3 = \rho \partial_t \partial_t u_3. \tag{1.15}$$

Note that this is a two-dimensional scalar equation, similar to (1.8) or (1.11).

The case for which $u_3 = 0$, while the other two displacement components are generally nonzero, (1.13), is called *inplane motion*. The initials P and SV are used to describe the two types of inplane motion, namely compressional and shear vertical, respectively. For this case (1.3) becomes

$$\tau_{\alpha\beta} = \lambda \partial_\gamma u_\gamma \delta_{\alpha\beta} + \mu(\partial_\alpha u_\beta + \partial_\beta u_\alpha), \tag{1.16}$$

and

$$\tau_{33} = \lambda \partial_\gamma u_\gamma. \tag{1.17}$$

The equation of motion remains (1.4); that is,

$$(\lambda + \mu)\partial_\alpha \partial_\beta u_\beta + \mu \partial_\beta \partial_\beta u_\alpha + \rho f_\alpha = \rho \partial_t \partial_t u_\alpha. \tag{1.18}$$

This last equation is a vector equation and contains two wave types, compressional and shear, whose character we explore shortly. It leads to problems of some complexity.

These two-dimensional equations are the principal models used. The scalar model, (1.14), allows us to solve complicated problems in detail without being overwhelmed by the size and length of the calculations needed, while the vector model, (1.18), allows us enough structure to indicate the complexity found in elastic-wave propagation.

1.1.3 Displacement Potentials

When (1.4)–(1.6) are used, a boundary condition, such as $\boldsymbol{t} = 0$, is relatively easy to implement. However, in problems in which there are few boundary conditions, it is often easier to cast the equations of motion into a simpler form and allow the boundary condition to become more complicated. One way to do this is to use the Helmholtz theorem (Phillips, 1933; Gregory, 1996) to express the particle displacement $\boldsymbol{u}$ as the sum of a scalar φ and a vector potential $\boldsymbol{\psi}$; that is,

$$\boldsymbol{u} = \nabla\varphi + \nabla \wedge \boldsymbol{\psi}, \qquad \nabla \cdot \boldsymbol{\psi} = 0. \tag{1.19}$$

The second condition is needed because $\boldsymbol{u}$ has only three components (the particular condition selected is not the only possibility). Assume $\boldsymbol{f} = 0$. Substituting these expressions into (1.6) gives

$$(\lambda + 2\mu)\nabla\left[\nabla^2\varphi - \left(1/c_L^2\right)\partial_t\partial_t\varphi\right] + \mu\nabla \wedge \left[\nabla^2\boldsymbol{\psi} - \left(1/c_T^2\right)\partial_t\partial_t\boldsymbol{\psi}\right] = 0. \tag{1.20}$$

The equation can be satisfied if

$$\nabla^2\varphi = \left(1/c_L^2\right)\partial_t\partial_t\varphi, \qquad c_L^2 = (\lambda + 2\mu)/\rho, \tag{1.21}$$

$$\nabla^2\boldsymbol{\psi} = \left(1/c_T^2\right)\partial_t\partial_t\boldsymbol{\psi}, \qquad c_T^2 = \mu/\rho. \tag{1.22}$$

The terms c_L and c_T are the compressional or longitudinal wavespeed, and shear or transverse wavespeed, respectively. The scalar potential describes a wave of compressional motion, which in the plane-wave case is longitudinal, while the vector potential describes a wave of shear motion, which in the plane-wave case is transverse. Knowing φ and $\boldsymbol{\psi}$, do we know $\boldsymbol{u}$ completely? Yes we do. Proofs of completeness, along with related references, are given in Achenbach (1973).

1.1.4 Energy Relations

The remaining conservation law of importance is the conservation of mechanical energy. Again assume $\boldsymbol{f} = 0$. This law can be derived directly from

(1.1)–(1.3) by taking the dot product of $\partial_t \boldsymbol{u}$ with (1.1). This gives, initially,

$$\partial_j \tau_{ji} \partial_t u_i - \rho(\partial_t \partial_t u_i)\partial_t u_i = 0. \qquad (1.23)$$

Forming the product $\tau_{kl}\epsilon_{kl}$, using (1.3), and making use of the decomposition $\partial_j u_i = \epsilon_{ji} + \omega_{ji}$, where $\omega_{ji} = (\partial_j u_i - \partial_i u_j)/2$, allows us to write (1.23) as

$$\tfrac{1}{2}\partial_t(\rho \partial_t u_i \partial_t u_i + \tau_{ki}\epsilon_{ki}) + \partial_k(-\tau_{ki}\partial_t u_i) = 0. \qquad (1.24)$$

The first two terms become the time rates of change of

$$\mathcal{K} = \tfrac{1}{2}\rho \partial_t u_k \partial_t u_k, \qquad \mathcal{U} = \tfrac{1}{2}\tau_{ij}\epsilon_{ij}. \qquad (1.25)$$

These are the kinetic and internal energy density, respectively. The remaining term is the divergence of the energy flux, $\boldsymbol{\mathcal{F}}$, where $\boldsymbol{\mathcal{F}}$ is given by

$$\mathcal{F}_j = -\tau_{ji}\partial_t u_i. \qquad (1.26)$$

Then (1.24) can be written as

$$\partial \mathcal{E}/\partial t + \nabla \cdot \boldsymbol{\mathcal{F}} = 0, \qquad (1.27)$$

where $\mathcal{E} = \mathcal{K} + \mathcal{U}$ and is the energy density. This is the differential statement of the conservation of mechanical energy. To better understand that the energy flux or power density is given by (1.26), consider an arbitrary region $\mathcal{R}$, with surface $\partial\mathcal{R}$. Integrating (1.27) over $\mathcal{R}$ and using Gauss' theorem gives

$$\frac{d}{dt}\int_{\mathcal{R}} \mathcal{E}(\boldsymbol{x}, t)\, dV = -\int_{\partial\mathcal{R}} \boldsymbol{\mathcal{F}} \cdot \hat{\boldsymbol{n}}\, dS. \qquad (1.28)$$

Therefore, as the mechanical energy decreases within $\mathcal{R}$, it radiates outward across the surface $\partial\mathcal{R}$ at a rate $\boldsymbol{\mathcal{F}} \cdot \hat{\boldsymbol{n}}$.

1.2 The Fourier and Laplace Transforms

All waves are transient in time. One useful representation of a transient waveform is its Fourier one. This representation imagines the transient signal decomposed into an infinite number of time-harmonic or frequency components. One important reason for the usefulness of this representation is that the transmitter, receiver, and the propagation structure usually respond differently to the different frequency components. The linearity of the problem ensures that we can work out the net propagation outcomes for each frequency component and then combine the outcomes to recreate the received signal.

The Fourier transform is defined as

$$\bar{u}(x,\omega) = \int_{-\infty}^{\infty} e^{i\omega t} u(x,t)\, dt. \tag{1.29}$$

The variable ω is complex. Its domain is such as to make the above integral convergent. Moreover, $\bar{u}$ is an analytic function within the domain of convergence, and once known, can be analytically continued beyond it.[1] The inverse transform is defined as

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \bar{u}(x,\omega)\, d\omega. \tag{1.30}$$

Thus we have represented u as a sum of harmonic waves $e^{-i\omega t}\bar{u}(x,\omega)$. *Note that there is a specific sign convention for the exponential term that we shall adhere to throughout the book.*

A closely related transform is the Laplace one. It is usually used for initial-value problems so that we imagine that for $t < 0$, $u(x,t)$ is zero. This is not essential and its definition can be extended to include functions whose t-dependence extends to negative values. This transform is defined as

$$\bar{u}(x,p) = \int_{0}^{\infty} e^{-pt} u(x,t)\, dt. \tag{1.31}$$

As with ω, p is a complex variable and its domain is such as to make $\bar{u}(x,p)$ an analytic function of p. The domain is initially defined as $\Re(p) > 0$, but the function can be analytically continued beyond this. Note that $p = -i\omega$ so that, when $t \in [0,\infty)$, $\Im(\omega) > 0$ gives the initial domain of analyticity for $\bar{u}(x,\omega)$. The inverse transform is given by

$$u(x,t) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} e^{pt} \bar{u}(x,p)\, dp, \tag{1.32}$$

where $\epsilon \geq 0$. The expressions for the inverse transforms, (1.30) and (1.32), are misleading. In practice we define the inverse transforms on contours that are designed to capture the physical features of the solution. A large part of this book will deal with just how those contours are selected. But, for the present, we shall work with these definitions.

[1] Analytic functions defined by contour integrals, including the case in which the contour extends to infinity, are discussed in Titchmarsh (1939) in a general way and in more detail by Noble (1988).

Consider the case of longitudinal strain. Imagine that at $t = -\infty$ a disturbance started with zero amplitude. Taking the Fourier transform of (1.8) gives

$$\left(d^2\bar{u}_1/dx_1^2\right) + k^2\bar{u}_1 = 0, \tag{1.33}$$

where $k = \omega/c_L$ and c_L is the compressional wavespeed defined in (1.21). The parameter k is called the wavenumber. Here (1.33) has solutions of the form

$$\bar{u}_1(x_1, \omega) = A(\omega)e^{\pm ikx_1}. \tag{1.34}$$

If we had sought a time-harmonic solution of the form

$$u_1(x_1, t) = \bar{u}_1(x_1, \omega)e^{-i\omega t}, \tag{1.35}$$

we should have gotten the same answer except that $e^{-i\omega t}$ would be present. In other words, taking the Fourier transform of an equation over time or seeking solutions that are time harmonic are two slightly different ways of doing the same operation.

For (1.35), it is understood that the real part can always be taken to obtain a real disturbance. Much the same happens in using (1.30). In writing (1.30) we implicitly assumed that $u(x, t)$ was real. That being the case, $\bar{u}(x, \omega) = \bar{u}^*(x, -\omega)$, where the superscript asterisk to the right of the symbol indicates the complex conjugate. From this it follows that

$$u(x, t) = \frac{1}{\pi}\Re\int_0^\infty e^{-i\omega t}\bar{u}(x, \omega)\, d\omega. \tag{1.36}$$

The advantage of this formulation of the inverse transform is that we may proceed with all our calculations by using an implied $e^{-i\omega t}$ and assuming ω is positive. The importance of this will become apparent in subsequent chapters. Now (1.36) can be regarded as a generalization of the taking of the real part of a time-harmonic wave (1.35).

Problem 1.1 Transform Properties

Check that $\bar{u}(x, \omega) = \bar{u}^*(x, -\omega)$ and derive (1.36) from (1.30). The reader may want to consult a book on the Fourier integral such as that by Papoulis (1962).

When the plus sign is taken, (1.35) is a time-harmonic, plane wave propagating in the positive x_1 direction. *We assume that the wavenumber k is positive, unless otherwise stated.* The wave propagates in the positive x_1 direction because the term $(kx_1 - \omega t)$ remains constant, and hence u_1 remains constant, only if x_1 increases as t increases. The speed with which the wave propagates is c_L. The term ω is the angular frequency or $2\pi f$, where f is the frequency. That is, at a fixed position, $1/f$ is the length of a temporal oscillation. Similarly, k, the wavenumber, is $2\pi/\lambda$, where λ, the wavelength, is the length of a spatial oscillation.

Note that if we combine two of these waves, labeled u_1^+ and u_1^-, each going in opposite directions, namely

$$u_1^+ = Ae^{i(kx_1-\omega t)}, \qquad u_1^- = Ae^{-i(kx_1+\omega t)}, \tag{1.37}$$

we get

$$u_1 = Ae^{-i\omega t} 2\cos(kx_1). \tag{1.38}$$

Taking the real part gives

$$u_1 = 2|A|\cos(\omega t + \alpha)\cos(kx_1). \tag{1.39}$$

We have taken A as complex so that α is its argument. This disturbance does not propagate. At a fixed x_1 the disturbance simply oscillates in time, and at a fixed t it oscillates in x_1. The wave is said to stand or is called a standing wave.

Problem 1.2 Fourier Transform

Continue with the case of longitudinal strain and consider the following boundary, initial-value problem. Unlike the previous discussion in which the disturbance began, with zero amplitude, at $-\infty$, here we shall introduce a disturbance that starts up at $t = 0^+$. Consider an elastic half-space, occupying $x_1 \geq 0$, subjected to a nonzero traction at its surface. The problem is one dimensional, and it is invariant in the other two so that (1.8), the equation for longitudinal strain, is the equation of motion. At $x_1 = 0$ we impose the boundary condition $\tau_{11} = -P_0\, e^{-\eta t} H(t)$, where $H(t)$ is the Heaviside step function and P_0 is a constant. As $x_1 \to \infty$ we impose the condition that any wave propagate outward in the positive x_1 direction. Why? Moreover, we ask that, for $t < 0$, $u_1(x_1, t) = 0$ and $\partial_t u_1(x_1, t) = 0$. Note that, using integration by parts, the Fourier transform, indicated by F, of the second time derivative is

$$F[\partial_t \partial_t u_1] = -\omega^2 \bar{u}_1(x_1, \omega) + i\omega u_1(x_1, 0^-) - \partial_t u_1(x_1, 0^-). \tag{1.40}$$

In deriving this expression we have integrated from $t = 0^-$ to ∞ so as to include any discontinuous behavior at $t = 0$. Taking the Fourier transform of (1.8) gives

(1.33). Show that the inverse transform of the stress component τ_{11} is given by

$$\tau_{11}(x_1, t) = \frac{P_0}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i(kx_1 - \omega t)}}{\omega + i\eta} \, d\omega. \tag{1.41}$$

In the course of making this step you will need to chose between the solutions to the transformed equation, (1.33). Why is the solution leading to (1.41) selected? Note that, if the disturbance is to decay with time, η must be positive. Next show that

$$\tau_{11}(x_1, t) = -P_0 H(t - x_1/c_L) e^{-\eta(t - x_1/c_L)}. \tag{1.42}$$

Explain how the conditions for convergence of the integral, as its contour is closed at infinity, give rise to the Heaviside function. Note how the sign conventions for the transform pair, by affecting where the inverse transform converges, give a solution that is causal.

Problem 1.3 Laplace Transform

Solve *Problem 1.2* by using the Laplace transform over time. Why select the solution e^{-px_1/c_L}? How does this relate to the demand that waves be outgoing at ∞?

The solution of *Problem 1.2* suggests how we shall define the Fourier transform over the spatial variable x. Suppose we have taken the temporal transform obtaining $\bar{u}(x, \omega)$. Then its Fourier transform over x is defined as

$$^*\bar{u}(k, \omega) = \int_{-\infty}^{\infty} e^{-ikx} u(x, t) \, dx, \tag{1.43}$$

and its inverse transform is

$$\bar{u}(x, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, {}^*\bar{u}(k, \omega) \, dk \, . \tag{1.44}$$

Again note the sign conventions for the transform pair. *This will remain the convention throughout the book*. Moreover, note that

$$u(x, t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(kx - \omega t)*}\bar{u}(k, \omega) \, d\omega \, dk \, . \tag{1.45}$$

This shows that quite arbitrary disturbances can be decomposed into a sum of time-harmonic, plane waves and *thereby indicates that the study of such waves is very central to the understanding of linear waves*.