CHAPTER ONE

Mathematical Background

This book assumes a background in the fundamentals of solid mechanics and the mechanical behavior of materials, including elasticity, plasticity, and friction. A previous book by the same authors[1] covers these topics in detail, including derivation or explanation of the most important concepts. It is beyond the scope of the current book to reproduce all of this important information.

In this chapter, the essential equations from this background are reproduced. This serves two purposes: to introduce the notation that will be used throughout the remaining chapters, and to list the principal background equations in one place. Frequent reference to the equations presented in this chapter will be made. However, it should be kept in mind that the full context for these equations is found in *Fundamentals of Metal Forming*.[1]

1.1 Notation

There are many alternate forms of notation used in solid mechanics and finite-element modeling. In some cases, it is clearer to use a form that has become a de facto standard in the area, even though such usage might not be rigorous. In other cases, there is no consensus on notation, so it is less confusing to be consistent with other equations.

In general, scalars are denoted by plain Roman or Greek letters, with or without subscripts or superscripts: a, A, α, t, T, a_1, a_{12}, ...

Vectors (whether physical or numerical ones, which are generalized one-dimensional arrays of numbers) are typically represented by lower-case or upper-case bold letters to emphasize the vector nature of the variable, with alternate notations used to refer to the components of the vector:

$$\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3 = |\mathbf{a}|\hat{\mathbf{g}} \leftrightarrow a_1, a_2, a_3 \leftrightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = [a_i] = [a] \leftrightarrow a_i, \qquad (1.1)$$

where $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$, are the Cartesian orthogonal unit vectors, $|\mathbf{a}|$ is the norm of vector **a**, and $\hat{\mathbf{g}}$ is the unit vector with direction **a**. The symbol $\leftrightarrow$ is used here in order to treat the differences in the forms rigorously. However, this convention will often be dropped and the various forms of such a quantity will be used interchangeably, depending on convenience and clarity.

[1] R. H. Wagoner and J.-L. Chenot, *Fundamentals of Metal Forming* (Wiley, New York, 1997).

Notation for tensors of rank higher than 1 ("vector") follows vector usage, although an attempt will be made to use bold upper-case letters when there is not a conventional usage of another symbol. Tensors are sometimes expressed in matrix form to illustrate the required manipulation:

$$\mathbf{A} \leftrightarrow [A] = [A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \leftrightarrow A_{ij}. \tag{1.2}$$

Of course, the matrix shown above need not correspond to any tensor (such as **A** shown at the left side of the chain). Several other common notations for matrices are as follows:

$$\begin{aligned} &\mathbf{A}^T \leftrightarrow [A]^T\text{: transpose of } \mathbf{A}, \\ &\mathbf{A}^{-1} \leftrightarrow [A]^{-1}\text{: inverse of } \mathbf{A}, \\ &\det(\mathbf{A}) = \det[A]\text{: determinant of } \mathbf{A}, \\ &\operatorname{trace}(\mathbf{A}) = \operatorname{trace}[A]\text{: trace of } \mathbf{A}, \\ &\mathbf{C} = \mathbf{AB} \leftrightarrow [C] = [A][B]\text{: matrix multiplication}, \\ &\mathbf{I} \leftrightarrow [I]\text{: identity matrix.} \end{aligned} \tag{1.3}$$

Each form can also be modified to show the subscript indices to emphasize the components.

The indicial form of matrix equations makes use of standard rules. Any repeated index within a term is a "dummy index," following Einstein's summation convention in which any repeated index is summed. Other indices are "free indices," which may independently adopt certain values. Two standard operators are used to complete indicial equations.

One is the Kronecker delta, with the property that

$$\begin{aligned} &\delta_{ij} = 0 \qquad \text{if } i \neq j, \\ &\delta_{ij} = 1 \qquad \text{if } i = j. \end{aligned} \tag{1.4}$$

The other is the permutation operator epsilon, with the property that

$$\begin{aligned} &\varepsilon_{ijk} = 0 \qquad \text{if } i = j, \text{ or } j = k, \text{ or } k = i, \\ &\varepsilon_{ijk} = 1 \qquad \text{if } ijk = 1, 2, 3, \text{ or } 2, 3, 1 \text{ or } 3, 1, 2, \\ &\varepsilon_{ijk} = -1 \qquad \text{if } ijk = 3, 2, 1, \text{ or } 1, 3, 2, \text{ or } 2, 1, 3. \end{aligned} \tag{1.5}$$

1.2 Stress

Throughout this book, reference to stress will always mean Cauchy stress, which relates real force intensities on planes and areas defined in a current deformation state. This standard stress measure is always symmetric by equilibrium considerations applied to a continuum and thus may be written as follows:

$$\boldsymbol{\sigma} \leftrightarrow [\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}. \tag{1.6}$$

The actual tractions, or stress vector, may be obtained from the stress tensor as follows:

$$\mathbf{T} = \boldsymbol{\sigma}\mathbf{n} \leftrightarrow [T_i] = \sum_j \sigma_{ij} n_j \leftrightarrow \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}. \tag{1.7}$$

The force acting on an elementary surface can be calculated similarly by

$$\mathrm{d}\mathbf{f} = \boldsymbol{\sigma}\mathrm{d}\mathbf{a} \leftrightarrow \mathrm{d}f_i = \sum_j \sigma_{ij} \mathrm{d}a_j \leftrightarrow \begin{bmatrix} \mathrm{d}f_1 \\ \mathrm{d}f_2 \\ \mathrm{d}f_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} \mathrm{d}a_1 \\ \mathrm{d}a_2 \\ \mathrm{d}a_3 \end{bmatrix}, \tag{1.8}$$

where $\mathrm{d}\mathbf{a} = |\mathrm{d}\mathbf{a}|\mathbf{n}$ is the elementary surface vector and $\mathbf{n}$ is the normal vector.

The principal stresses are obtained as the eigenvalues λ of $\boldsymbol{\sigma}$ in the usual way:

$$\boldsymbol{\sigma}\mathbf{n} = \lambda\mathbf{n} \leftrightarrow \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \lambda n_1 \\ \lambda n_2 \\ \lambda n_3 \end{bmatrix}, \tag{1.9}$$

or with indicial form:

$$\begin{bmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0. \tag{1.10}$$

This is solved by noting that the determinant of the tensor on the left-hand side must be identically equal to zero (if $\mathbf{n}$ is not to be a null vector):

$$\det(\boldsymbol{\sigma}) = 0 \leftrightarrow \det \begin{bmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{bmatrix} = 0, \tag{1.11}$$

where the result of this calculation gives a cubic equation, with real roots λ_i corresponding to the principal stresses (eigenvalues):

$$\lambda^3 - J_1\lambda^2 - J_2\lambda - J_3 = 0, \tag{1.12}$$

where J_1 is the first stress invariant,

$$J_1 = \mathrm{trace}(\boldsymbol{\sigma}) = \sigma_{11} + \sigma_{22} + \sigma_{33}, \tag{1.13}$$

J_2 is the second stress invariant (quadratic invariant),

$$J_2 = -(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}) + \sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2, \tag{1.14}$$

and J_3 is the third stress invariant,

$$J_3 = \det(\boldsymbol{\sigma}) = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix}. \tag{1.15}$$

For the numerical formulation of elastoplastic constitutive equations, it is frequently convenient to perform an additive decomposition of the stress tensor to obtain the spherical or pressure part (which is insensitive to plastic deformation) and the deviatoric part. The hydrostatic pressure is taken to be:

$$p = -\sigma_p = -\frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} = -\frac{J_1}{3}, \qquad (1.16)$$

which allows further decomposition to obtain the deviatoric part:

$$\boldsymbol{\sigma} = \sigma_p \mathbf{I} + \mathbf{s} = (-p)\mathbf{I} + \mathbf{s}, \qquad (1.17)$$

where $\mathbf{s}$ is the deviatoric stress tensor.

1.3 Strain

The work-conjugate (actually power-conjugate) quantity to the Cauchy stress is the rate of deformation ($\mathbf{D}$), which may be referred to without ambiguity as the strain rate $\dot{\varepsilon}$ (epsilon with overdot). In nearly every instance throughout this book, an updated Lagrangian formulation will be used, in which the preceding and current steps are considered sufficiently close that infinitesimal deformation theory may be used, such that, for example, $\Delta\varepsilon = \dot{\varepsilon}\Delta t$. Thus, there will be no distinction between power and work formulations. With this in mind, a few important equations may be presented, starting with the Lagrangian view of continuum deformation.

The coordinate vector is

$$\mathbf{x} = \chi(\mathbf{X}, t), \qquad (1.18)$$

where each material point is labeled by its position $\mathbf{X}$ at some time t_0, and its current position at time t is given by $\mathbf{x}$.

The material velocity is

$$\mathbf{v} = \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{X}}. \qquad (1.19)$$

The material acceleration is

$$\gamma = \left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{X}}. \qquad (1.20)$$

The displacement vector is

$$\mathbf{U}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}. \qquad (1.21)$$

In the updated Lagrangian sense, $\mathbf{X}$ represents the position of a material point at the previous time step, which is considered only infinitesimally removed from the current position of the same material element, $\mathbf{x}$.

The deformation gradient defines the transformation between corresponding material vectors at the two instants,

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}\mathrm{d}\mathbf{X} = \mathbf{F}\,\mathrm{d}\mathbf{X}, \qquad (1.22)$$

with the component form

$$\begin{bmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} \mathrm{d}X_1 \\ \mathrm{d}X_2 \\ \mathrm{d}X_3 \end{bmatrix} \leftrightarrow \mathrm{d}x_i = \sum_j \frac{\partial x_i}{\partial X_j}\mathrm{d}X_j = \sum_j F_{ij}\mathrm{d}X_j. \qquad (1.23)$$

It may be shown that the determinant of **F**, also called the Jacobian of the transformation, relates the initial differential volume dvol_0 (or density ρ_0) at a point with the corresponding one dvol (or ρ) after deformation:

$$\frac{\mathrm{dvol}}{\mathrm{dvol}_0} = \frac{\rho}{\rho_0} = J = \det(\mathbf{F}). \tag{1.24}$$

A similar transformation is written for the relative displacements of the head and tail of such a vector:

$$\mathrm{d}\mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \mathbf{J}\,\mathrm{d}\mathbf{X}. \tag{1.25}$$

With the indicial forms we have

$$\begin{bmatrix} \mathrm{d}u_1 \\ \mathrm{d}u_2 \\ \mathrm{d}u_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{bmatrix} \begin{bmatrix} \mathrm{d}X_1 \\ \mathrm{d}X_2 \\ \mathrm{d}X_3 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} \begin{bmatrix} \mathrm{d}X_1 \\ \mathrm{d}X_2 \\ \mathrm{d}X_3 \end{bmatrix}. \tag{1.26}$$

$$\leftrightarrow \mathrm{d}u_i = \sum_j \frac{\partial u_i}{\partial X_j}\mathrm{d}X_j = \sum_j J_{ij}\mathrm{d}X_j$$

J is called the displacement gradient matrix, or Jacobian matrix, which can be written in terms of **F** with the help of Eq. (1.21):

$$\mathbf{J} = \mathbf{F} - \mathbf{I}. \tag{1.27}$$

In order to ignore pure rotations of a vector, the stretch of a material element is considered, starting from **F**:

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{X}^T\mathbf{C}\,\mathrm{d}\mathbf{X}, \tag{1.28}$$

with

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}, \quad C_{ij} = \sum_k \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}, \tag{1.29}$$

where ds is the final length of such an element or vector, and **C** is known as the deformation tensor, or the Cauchy deformation tensor.

Whereas **C** transforms the length of a vector from one state to another, it is most frequently of interest to focus on the change of length, in which case the strain tensor **E** is used:

$$\mathrm{d}s^2 - \mathrm{d}S^2 = \mathrm{d}\mathbf{X}^T(2\mathbf{E})\,\mathrm{d}\mathbf{X}. \tag{1.30}$$

Taking into account Eq. (1.28), we get

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}), \tag{1.31}$$

the components of which are

$$E_{ij} = \frac{1}{2}\left(\sum_k \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right), \tag{1.32}$$

and with the help of Eq. (1.27) we can also write

$$E = \frac{1}{2}(\mathbf{J} + \mathbf{J}^T + \mathbf{J}^T\mathbf{J}). \tag{1.33}$$

The factor of 1/2 is used conventionally such that the infinitesimal strain components (which are historically older) become the small limit of **E**.

For small deformation and rotation (i.e., where the components of J are much less than one, and can be considered in the limit of approaching zero), it may be seen that the last term of Eq. (1.33) is vanishingly small relative to the first term. Elimination of the second-order term produces the definition of the small-strain tensor:

$$\varepsilon = \frac{1}{2}(\mathbf{J} + \mathbf{J}^T) \quad \text{or } \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.34}$$

The small strain is clearly the symmetric part of **J**, whereas the antisymmetric part corresponds to the rigid-body rotation:

$$\omega = \frac{1}{2}(\mathbf{J} - \mathbf{J}^T) \quad \text{or } \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right). \tag{1.35}$$

Although there is no formal distinction between infinitesimal displacements and velocities (aside from a homogeneous factor of dt), conventional notation is often based on rate or velocity forms. In this case, the relative velocity d**v** of the head to tail of a vector d**x** is related to d**x** as follows[2]:

$$\mathrm{d}\mathbf{v} = \mathbf{L}\,\mathrm{d}\mathbf{x}, \tag{1.36}$$

where **L** is defined by

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}, \quad L_{ij} = \frac{\partial v_i}{\partial x_j}, \tag{1.37}$$

and **L** is the velocity gradient or the spatial gradient of velocity. **L** may be decomposed additively, analogous to the decomposition of **J** (because **L** is simply $\dot{\mathbf{J}}$) for infinitesimal differences between the start and end of deformation:

$$\mathbf{L} = \dot{\varepsilon} + \dot{\omega}. \tag{1.38}$$

Here $\dot{\varepsilon}$, often denoted by **D**, is the strain rate, or rate of deformation tensor:

$$2\dot{\varepsilon} = \mathbf{L} + \mathbf{L}^T, \quad \dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{1.39}$$

and $\dot{\omega}$ is the spin tensor:

$$2\dot{\omega} = \mathbf{L} - \mathbf{L}^T, \quad \dot{\omega}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right). \tag{1.40}$$

[2] Note that the lower case has been used for d**x**, even though this is a material vector at the beginning of the deformation step. Because the focus is on infinitesimal steps, the distinction between d**x** and d**X** is lost in this context.

For small strain, denoted here by Δ, the relationship between $\Delta\varepsilon$ and $\dot{\varepsilon}$, or $\Delta\omega$ and $\dot{\omega}$, is thus:

$$\Delta\varepsilon = \dot{\varepsilon}\Delta t, \tag{1.41}$$

$$\Delta\omega = \dot{\omega}\Delta t. \tag{1.42}$$

It is sometimes useful to use relationships among the various deformation measures:

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} \quad \text{or } \dot{\mathbf{F}} = \mathbf{LF}, \tag{1.43}$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}), \quad E_{ij} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_J} - \delta_{ij}\right). \tag{1.44}$$

It is also useful to perform a polar decomposition of the deformation gradient, as follows:

$$\mathbf{F} = \mathbf{RU}, \tag{1.45}$$

where $\mathbf{R}$ is orthogonal and called the rotational operator and where $\mathbf{U}$ is symmetric positive definite and is called the right stretch tensor. $\mathbf{R}$ is often used to estimate the rigid-body rotation of a large deformation, whereas $\mathbf{U}$ is used to find the stretch ratios for a large deformation:

$$\mathbf{U}^2 = \mathbf{C} = (2\mathbf{E} + \mathbf{I}), \quad \lambda_i = U_i = \sqrt{C_i} = \sqrt{2E_i + 1}. \tag{1.46}$$

Here λ_i is the ith stretch ratio corresponding to the ith principal value (eigenvalue) of $\mathbf{U}$. Note that raising a tensor to a power signifies a tensor with the same principal directions but with principal values raised to that power.

1.4 Mechanical Principles

There are many alternate, but equivalent, ways to formulate the mechanical equations governing continuum motion. The continuity equation states that mass cannot be lost or gained, and it implies that velocity fields must be well behaved:

$$\frac{d\rho}{dt} + \rho\,\mathrm{div}(\mathbf{v}) = 0, \quad \frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\mathbf{v}) = 0. \tag{1.47}$$

Similarly, Newton's laws must be obeyed for each material element. Including dynamic (inertial) and static effects internally, and gravity as an external body force, we can write the equation of motion as

$$\rho\gamma = \mathrm{div}(\boldsymbol{\sigma}) + \rho\mathbf{g}. \tag{1.48}$$

For the cases that dominate the examples in this book, only static equilibrium need be considered, and gravity forces may be neglected as much smaller than other forces:

$$\mathrm{div}(\boldsymbol{\sigma}) = 0. \tag{1.49}$$

By considering a virtual displacement field $\delta\mathbf{u}$ (which is infinitesimal and has the property that $\delta\mathbf{u}$ is zero wherever the displacement is specified), we can equate the internal work absorbed by the deformation of the material to the external work done by outside forces acting on the body:

$$\int_{\partial\Omega} \mathbf{T}^d \delta\mathbf{u}\,\mathrm{d}S + \int_{\Omega} \rho\mathbf{g}\delta\mathbf{u}\,\mathrm{d}V = \int_{\Omega} \rho\gamma\delta\mathbf{u}\,\mathrm{d}V + \int_{\Omega} \boldsymbol{\sigma}\delta\varepsilon\,\mathrm{d}V,$$

$$\text{external work increment} = \text{internal work increment} \tag{1.50}$$

where $\mathbf{T}^d$ is defined on the whole boundary $\partial\Omega$ of Ω. It is equal to the external stress vector where it is prescribed, and it is equal to zero elsewhere; $\delta\mathbf{u}$ is any virtual displacement equal to zero on the part of $\partial\Omega$ where the displacement is imposed.

As before, if static equilibrium is sought and gravitational forces may be neglected as second order, this reduces to

$$\int_{\partial\Omega} \mathbf{T}^d \delta\mathbf{u}\,\mathrm{d}S = \int_{\Omega} \boldsymbol{\sigma}\delta\varepsilon\,\mathrm{d}V.$$

$$\text{external work increment} = \text{internal work increment} \tag{1.51}$$

Variational principles may be used to establish functional formulations whenever the virtual work principle can be integrated exactly. Then, instead of solving for a root of a function where the net force equals zero [e.g., the function can be the left-hand side of Eq. (1.50) less the right-hand side of Eq. (1.51)], we seek the minimum of a functional, generally homogeneous to the net work, or to the rate of work.

Although it is always possible to derive the virtual work statement from the functional (by differentiation), the converse is not always possible. Where the variational principle or functional exists, it ensures that the stiffness matrix is symmetric, which has advantages for numerical solution. Thus, when such a principle can be written, it is often useful to do so. Although there is no convenient elastoplastic functional, both elastic and purely plastic or viscoplastic functionals can be derived. Examples of such functionals are shown as follows, under the appropriate constitutive equations.

1.5 Elasticity

Hooke's law may be written simply as follows:

$$\boldsymbol{\sigma} = \mathbf{c} : \varepsilon, \qquad \text{or } \varepsilon = \mathbf{S} : \boldsymbol{\sigma}. \tag{1.52}$$

With the components notation it is also written as

$$\sigma_{ij} = \sum_{k,l} c_{ijkl}\varepsilon_{kl}, \qquad \text{or } \varepsilon_{ij} = \sum_{k,l} S_{ijkl}\sigma_{kl}, \tag{1.53}$$

where $\mathbf{c}$ is known as the elastic constant tensor (often denoted by $\mathbf{D}$) and $\mathbf{S}$ represents the compliance tensor.

For an isotropic material, $\mathbf{c}$ and $\mathbf{S}$ take special forms ensuring that the material has the same properties in every direction. The terms in $\mathbf{c}$ and $\mathbf{S}$ are often written in terms of conventional elastic constants, as follows:

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{1.54}$$

$$S_{ijkl} = \frac{1+\nu}{E}\delta_{ij}\delta_{kl} - \frac{\nu}{E}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{1.55}$$

where the conventional elastic constants λ, μ are the Lamé coefficients, E is the Young modulus, and ν is the Poisson coefficient.

Using Eq. (1.54), we write Hooke's equation in the usual form:

$$\boldsymbol{\sigma} = \lambda\theta\mathbf{I} + 2\mu\boldsymbol{\varepsilon}, \tag{1.56}$$

where $\theta = \sum_i \varepsilon_{ii}$ is the dilatation. The component form is obviously

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij}. \tag{1.57}$$

If Eq. (1.56) is inverted, we get

$$\boldsymbol{\varepsilon} = \frac{3\nu}{E}p\mathbf{I} + \frac{1+\nu}{E}\boldsymbol{\sigma} \tag{1.58}$$

or

$$\varepsilon_{ij} = \frac{3\nu}{E}p\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}. \tag{1.59}$$

Other elastic constants are also used commonly and are related as follows:

$$\mu = G = \frac{E}{2(1+\nu)}, \tag{1.60}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1+2\nu)} = \frac{2\mu\nu}{1-2\nu}. \tag{1.61}$$

For the bulk modulus,

$$B = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)} = \kappa, \tag{1.62}$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)} = \frac{3B-2\mu}{2(3B+\mu)}, \tag{1.63}$$

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{9\mu B}{3B+\mu}. \tag{1.64}$$

For this material constitutive equation (i.e., an isotropic, linearly elastic material), a functional exists and may be written as follows:

$$\Pi(\boldsymbol{\varepsilon}) = \int_\Omega \left(\frac{1}{2}\lambda\theta^2 + \mu\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}\right)\mathrm{d}V - \int_{\partial\Omega} \mathbf{T}^d\mathbf{u}\,\mathrm{d}S. \tag{1.65}$$

Here $\mathbf{T}^d$ is defined on the whole boundary $\partial\Omega$ of Ω. It is equal to the external stress vector where it is prescribed, and it is equal to zero elsewhere; $\delta\mathbf{u}$ is any virtual displacement equal to zero on the part of $\partial\Omega$ where the displacement is imposed.

1.6 Plasticity

Unless otherwise stated, in this book we consider only flow theory plasticity laws, which obey the following principles.

$f(\boldsymbol{\sigma}) < 0$: no plastic deformation,

$f(\boldsymbol{\sigma}) = 0$: plastic deformation is possible,

$f(\boldsymbol{\sigma}) > 0$: is forbidden.

- $f(\boldsymbol{\sigma})$ is a yield function with the following properties:
  - There is no plastic deformation in the elastic (or rigid) region enclosing $\boldsymbol{\sigma} = 0$, which is defined by

$$f(\boldsymbol{\sigma}) < 0, \quad \text{or} \left(f(\boldsymbol{\sigma}) = 0 \quad \text{and} \quad \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} < 0 \right). \tag{1.66}$$

  - The plastic region corresponds to

$$f(\boldsymbol{\sigma}) = 0 \quad \text{and} \quad \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} \geq 0. \tag{1.67}$$

  - The surface defined in the stress space by $f(\boldsymbol{\sigma}) = 0$ is convex.
- The yield surface changes size but not shape (isotropic hardening).
- The normality condition (associated flow) holds for plastic loading:

$$\dot{\varepsilon} = \dot{\lambda}^p \frac{\partial f}{\partial \boldsymbol{\sigma}}, \quad \dot{\varepsilon}_{ij} = \dot{\lambda}^p \frac{\partial f}{\partial \sigma_{ij}}, \qquad \dot{\lambda}^p > 0, \tag{1.68}$$

or

$$\mathrm{d}\varepsilon = \mathrm{d}\lambda^p \frac{\partial f}{\partial \boldsymbol{\sigma}}, \quad \mathrm{d}\varepsilon_{ij} = \mathrm{d}\lambda^p \frac{\partial f}{\partial \sigma_{ij}}, \qquad \mathrm{d}\lambda^p > 0. \tag{1.69}$$

- Plastic flow is time independent (unless viscoplasticity is noted).

The von Mises yield function may be expressed in stress components as follows:

$$\begin{aligned} f(\boldsymbol{\sigma}) &= \frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] - \bar{\sigma}_0^2 \\ &= \frac{1}{2}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6\sigma_{12}^2 + 6\sigma_{23}^2 + 6\sigma_{31}^2] - \bar{\sigma}_0^2. \end{aligned} \tag{1.70}$$

If the deviatoric stress tensor $\mathbf{s}$ is used, we obtain the equivalent expression,

$$f(\mathbf{s}) = \frac{3}{2} \sum_{i,j} s_{ij}^2 - \bar{\sigma}_0^2, \tag{1.71}$$

which is also

$$f(\mathbf{s}) = J_2' - \bar{\sigma}_0^2, \tag{1.72}$$

where J_2' is the second invariant of the deviatoric stress tensor [see Eq. (1.14)] and $\bar{\sigma}_0$ is the yield stress in tension. We also define σ_i and s_i as the principal values of the stress tensor and deviatoric stress tensor, respectively.

It is convenient to define the effective stress (or equivalent stress) as the tensile stress corresponding to any state of stress by means of a yield surface passing through the state of stress:

$$\bar{\sigma} = f(\boldsymbol{\sigma}). \tag{1.73}$$

For the von Mises flow, the effective stress takes the usual form:

$$\begin{aligned} \bar{\sigma} &= \left\{ \frac{1}{2}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6\sigma_{12}^2 + 6\sigma_{23}^2 + 6\sigma_{31}^2] \right\}^{1/2} \\ &= \left\{ \frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \right\}^{1/2} = \left(\frac{3}{2} \sum_{i,j} s_{ij}^2 \right)^{1/2}. \end{aligned} \tag{1.74}$$