HARMONIC MAPPINGS IN THE PLANE

Harmonic mappings in the plane are univalent complex-valued harmonic functions of a complex variable. Conformal mappings are a special case where the real and imaginary parts are conjugate harmonic functions, satisfying the Cauchy–Riemann equations. Harmonic mappings were studied classically by differential geometers because they provide isothermal (or conformal) parameters for minimal surfaces. More recently they have been actively investigated by complex analysts as generalizations of univalent analytic functions, or conformal mappings. Many classical results of geometric function theory extend to harmonic mappings, but basic questions remain unresolved.

This book is the first comprehensive account of the theory of planar harmonic mappings, treating both the generalizations of univalent analytic functions and the connections with minimal surfaces. Essentially self-contained, the book contains background material in complex analysis and a full development of the classical theory of minimal surfaces, including the Weierstrass–Enneper representation. It is designed to introduce nonspecialists to a beautiful area of complex analysis and geometry.

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CAMBRIDGE TRACTS IN MATHEMATICS

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32 Avenue of the Americas, New York NY 10013-2473, USA

Cambridge University Press is part of the University of Cambridge.

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www.cambridge.org
Information on this title: www.cambridge.org/9780521641210

First published 2004, 2011
Second Edition 2012
Reprinted 2013

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Duren, Peter L., 1935–
Harmonic mappings in the plane / Peter Duren.
p. cm. – (Cambridge tracts in mathematics ; 156)
Includes bibliographical references and index.
ISBN 0-521-64121-7
1. Harmonic maps. I. Title. II. Series.
QA614.73.D87 2004
514′.74 – dc22 2003056516

ISBN 978-0-521-64121-0 Hardback

Dedicated to the memory of
Glenn Schober
(1938–1991)

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Preface

Harmonic mappings in the plane are univalent complex-valued harmonic functions whose real and imaginary parts are not necessarily conjugate. In other words, the Cauchy–Riemann equations need not be satisfied, so the functions need not be analytic. Although harmonic mappings are natural generalizations of conformal mappings, they were studied originally by differential geometers because of their natural role in parametrizing minimal surfaces. Only in the mid-1980s did harmonic mappings begin to attract widespread interest among complex analysts. The catalyst was a landmark paper by James Clunie and Terry Sheil-Small in 1984, pointing out that many of the classical results for conformal mappings have clear analogues for harmonic mappings. Since that time the subject has developed rapidly, although a number of basic problems remain unresolved. This book is an attempt to make this beautiful material accessible to a wider mathematical public.

Most of the book concerns harmonic mappings in the plane, but there are occasional excursions into higher dimensions, if only to provide counterexamples. As a general rule, the rich structure of theory in the plane does not extend to higher-dimensional space. In many instances, the properties of *analytic* univalent functions serve as models for generalizations to harmonic mappings, but other results are peculiar to analytic functions and do not extend to more general harmonic mappings. On the other hand, some results for harmonic mappings have no counterpart for conformal mappings. This is particularly true of the connections with minimal surfaces, which are developed in the final two chapters.

The book is dedicated to my collaborator and close friend Glenn Schober. I began writing it a few months before Glenn's untimely death in 1991 and had the benefit of discussing its contents with him as the project took shape. It would have been a better book if Glenn could have written it with me. In any event, it certainly reflects ideas and insights gained through our long association.

I am also grateful to Harold Shapiro, Walter Hengartner, and Terry Sheil-Small for teaching me essential things about harmonic mappings. Many people read and criticized early drafts of the book. First and foremost, Richard Laugesen went through much of the manuscript with a fine-toothed comb, spotted errors and ambiguities, and suggested many improvements. His generous help and constant encouragement were invaluable. Others who read and criticized portions of the manuscript were Walter Hengartner, Paul Greiner, John Pfaltzgraff, Željko Čučković, Michael Dorff, and Dmitry Khavinson. Their comments were helpful and are much appreciated. Paul Greiner also assisted in producing the figures drawn with *Mathematica*. Marcin Bownik was a great help in extracting the figures from the computer and preparing them for publication.

Finally, I would like to thank Lauren Cowles, David Tranah, and others at Cambridge University Press for expert advice and technical assistance, and for amazing patience with a long overdue manuscript.

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