Advanced Transport Phenomena

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK http: //www.cup.cam.ac.uk
40 West 20th Street, New York, NY 10011-4211, USA http: //www.cup.org
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1999

Printed in the United States of America

Typeset in Gill Sans and Times Roman LaTeX 2_ε [TB]

A catalog record for this book is available from the British Library

Library of Congress Cataloging-in-Publication Data

Slattery, John Charles, 1932–
Advanced transport phenomena / John C. Slattery.
p. cm. – (Cambridge series in chemical engineering)
ISBN 0-521-63203-X (hb). – ISBN 0-521-63565-9 (pb)
1. Transport theory. 2. Chemical engineering. I. Title.
II. Series.
TP156.T7S57 1999
660′.2842 – dc21 98-44872
CIP

ISBN 0 521 63203X hardback
ISBN 0 521 635659 paperback

Contents

1

Kinematics

THIS ENTIRE CHAPTER IS INTRODUCTORY in much the same way as Appendix A is. In Appendix A, I introduce the mathematical language that I shall be using in describing physical problems. In this chapter, I indicate some of the details involved in representing from the continuum point of view the motions and deformations of real materials. This chapter is important not only for the definitions introduced, but also for the viewpoint taken in some of the developments. For example, the various forms of the transport theorem will be used repeatedly throughout the text in developing differential equations and integral balances from our basic postulates.

Perhaps the most difficult point for a beginner is to properly distinguish between the continuum model for real materials and the particulate or molecular model. We can all agree that the most factually detailed picture of real materials requires that they be represented in terms of atoms and molecules. In this picture, mass is distributed discontinuously throughout space; mass is associated with the protons, neutrons, electrons, . . . , which are separated by relatively large voids. In the continuum model for materials, mass is distributed continuously through space, with the exception of surfaces of discontinuity, which represent phase interfaces or shock waves.

The continuum model is less realistic than the particulate model but far simpler. For many purposes, the detailed accuracy of the particulate model is unnecessary. To our sight and touch, mass appears to be continuously distributed throughout the water that we drink and the air that we breathe. The problem is analogous to a study of traffic patterns on an expressway. The speed and spacings of the automobiles are important, but we probably should not worry about whether the automobiles have four, six, or eight cylinders.

This is not to say that the particulate theories are of no importance. Information is lost in a continuum picture. It is only through the use of statistical mechanics that a complete a priori prediction about the behavior of the material can be made. I will say more about this in the next chapter.

1.1 Motion

My goal in this book is to lay the foundation for understanding a wide variety of operations employed in the chemical and petroleum industries. To be specific, consider the extrusion of

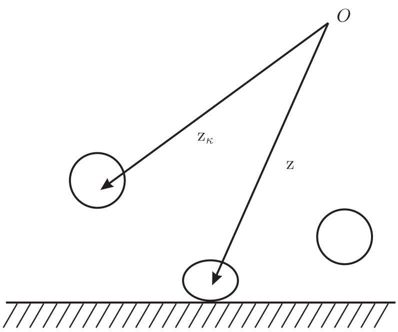

Figure 1.1.0-1. A rubber ball in three configurations as it strikes a wall and rebounds. The particle that was in the position $\mathbf{z}_\kappa$ in the reference configuration is in the position $\mathbf{z}$ at time t.

a molten polymer to produce a fiber, catalytic cracking in a fluidized reaction, the production of oil and gas from a sandstone reservoir, or the flow of a coal slurry through a pipeline. One important feature that these operations have in common is that at least some of the materials concerned are undergoing deformation and flow.

How might we describe a body of material as it deforms? Figure 1.1.0-1 shows a rubber ball in three configurations as it strikes a wall and rebounds. How should we describe the deformation of this rubber body from its original configuration as a sphere? How should velocity be defined in order to take into account that it must surely vary as a function of position within the ball as well as time as the ball reaches the wall and begins to deform? We need a mathematical description for a body that allows us to describe where its various components go as functions of time.

Let's begin by rather formally defining a body to be set, any element ζ of which is called a particle or a *material particle*. A material particle is a *primitive*, in the sense that it is not defined but its properties are described. I will give an experimentally oriented description of a material particle a little later in this section. Meanwhile, be careful not to confuse a material particle with a molecule. Molecules play no role in continuum mechanics; they are introduced in the context of the other model for real materials – statistical mechanics.

A one-to-one continuous mapping of this set of material particles onto a region of the space E studied in elementary geometry exists and is called a *configuration* of the body:

$$z = X(\zeta) \tag{1.1.0-1}$$

$$\zeta = X^{-1}(z) \tag{1.1.0-2}$$

The point $z = X(\zeta)$ of E is called the place occupied by the particle ζ, and $\zeta = X^{-1}(z)$, the particle whose place in E is z.

It is completely equivalent to describe the configuration of a body in terms of the position vector $\mathbf{z}$ of the point z with respect to the origin O (Section A.1.2):

$$\mathbf{z} = \chi(\zeta) \tag{1.1.0-3}$$

$$\zeta = \chi^{-1}(\mathbf{z}) \tag{1.1.0-4}$$

Here χ^{-1} indicates the inverse mapping of $\boldsymbol{\chi}$. With an origin O having been defined, it is unambiguous to refer to $\mathbf{z} = \boldsymbol{\chi}(\zeta)$ as the place occupied by the particle ζ and $\zeta = \chi^{-1}(\mathbf{z})$ as the particle whose place is $\mathbf{z}$.

In what follows, I choose to refer to points in E by their position vectors relative to a previously defined origin O.

A *motion* of a body is a one-parameter family of configurations; the real parameter t is time. We write

$$\mathbf{z} = \boldsymbol{\chi}(\zeta, t) \tag{1.1.0-5}$$

and

$$\zeta = \chi^{-1}(\mathbf{z}, t) \tag{1.1.0-6}$$

I have introduced the material particle as a primitive concept, without definition but with a description of its attributes. A set of material particles is defined to be a body; there is a one-to-one continuous mapping of these particles onto a region of the space E in which we visualize the world about us. But clearly we need a link with what we can directly observe.

Whereas the body B should not be confused with any of its spatial configurations, it is available to us for observation and study only in these configurations. We will describe a material particle by its position in a *reference configuration* $\boldsymbol{\kappa}$ of the body. This reference configuration may be, but need not be, one actually occupied by the body in the course of its motion. The place of a particle in $\boldsymbol{\kappa}$ will be denoted by

$$\mathbf{z}_\kappa = \boldsymbol{\kappa}(\zeta) \tag{1.1.0-7}$$

The particle at the place $\mathbf{z}_\kappa$ in the configuration $\boldsymbol{\kappa}$ may be expressed as

$$\zeta = \kappa^{-1}(\mathbf{z}_\kappa) \tag{1.1.0-8}$$

The motion of a body is described by

$$\begin{aligned}\mathbf{z} &= \boldsymbol{\chi}(\zeta, t) \\ &= \boldsymbol{\chi}_\kappa(\mathbf{z}_\kappa, t) \\ &\equiv \boldsymbol{\chi}(\kappa^{-1}(\mathbf{z}_\kappa), t)\end{aligned} \tag{1.1.0-9}$$

Referring to Figure 1.1.0-1, we find that the particle that was in the position $\mathbf{z}_\kappa$ in the reference configuration is at time t in the position $\mathbf{z}$. This expression defines a family of *deformations* from the reference configuration. The subscript $\ldots_\kappa$ is to remind you that the form of $\boldsymbol{\chi}_\kappa$ depends upon the choice of reference configuration $\boldsymbol{\kappa}$.

The position vector $\mathbf{z}_\kappa$ with respect to the origin O may be written in terms of its rectangular Cartesian coordinates:

$$\mathbf{z}_\kappa = z_{\kappa i} \mathbf{e}_i \tag{1.1.0-10}$$

The $z_{\kappa i}$ $(i = 1, 2, 3)$ are referred to as the *material coordinates* of the material particle ζ. They locate the position of ζ relative to the origin O, when the body is in the reference configuration $\boldsymbol{\kappa}$. In terms of these material coordinates, we may express (1.1.0-9) as

$$\begin{aligned}\mathbf{z} &= \boldsymbol{\chi}_\kappa (\mathbf{z}_\kappa, t) \\ &= \hat{\boldsymbol{\chi}}_\kappa (z_{\kappa 1}, z_{\kappa 2}, z_{\kappa 3}, t)\end{aligned} \tag{1.1.0-11}$$

Let A be any quantity: scalar, vector, or tensor. We shall have occasion to talk about the time derivative of A following the motion of a particle. We define

$$\begin{aligned}\frac{d_{(m)}A}{dt} &\equiv \left(\frac{\partial A}{\partial t}\right)_{\zeta} \\ &\equiv \left(\frac{\partial A}{\partial t}\right)_{\mathbf{z}_\kappa} \\ &\equiv \left(\frac{\partial A}{\partial t}\right)_{z_{\kappa 1}, z_{\kappa 2}, z_{\kappa 3}}\end{aligned} \tag{1.1.0-12}$$

We refer to the operation $d_{(m)}/dt$ as the *material derivative* [or substantial derivative (Bird et al. 1960, p. 73)]. For example, the *velocity vector* $\mathbf{v}$ represents the time rate of change of position of a material particle:

$$\begin{aligned}\mathbf{v} &\equiv \frac{d_{(m)}\mathbf{z}}{dt} \\ &\equiv \left[\frac{\partial \boldsymbol{\chi}(\zeta, t)}{\partial t}\right]_{\zeta} \\ &\equiv \left[\frac{\partial \boldsymbol{\chi}_\kappa(\mathbf{z}_\kappa, t)}{\partial t}\right]_{\mathbf{z}_\kappa} \\ &\equiv \left[\frac{\partial \hat{\boldsymbol{\chi}}_\kappa(z_{\kappa 1}, z_{\kappa 2}, z_{\kappa 3}, t)}{\partial t}\right]_{z_{\kappa 1}, z_{\kappa 2}, z_{\kappa 3}}\end{aligned} \tag{1.1.0-13}$$

We are involved with several derivative operations in the chapters that follow. Bird et al. (1960, p. 73) have suggested some examples that serve to illustrate the differences.

The partial time derivative $\partial c/\partial t$ Suppose we are in a boat that is anchored securely in a river, some distance from the shore. If we look over the side of our boat and note the concentration of fish as a function of time, we observe how the fish concentration changes with time at a fixed position in space:

$$\begin{aligned}\frac{\partial c}{\partial t} &\equiv \left(\frac{\partial c}{\partial t}\right)_{\mathbf{z}} \\ &\equiv \left(\frac{\partial c}{\partial t}\right)_{z_1, z_2, z_3}\end{aligned}$$

The material derivative $d_{(m)}c/dt$ Suppose we pull up our anchor and let our boat drift along with the river current. As we look over the side of our boat, we report how the concentration of fish changes as a function of time while following the water (the material):

$$\frac{d_{(m)}c}{dt} = \frac{\partial c}{\partial t} + \nabla c \cdot \mathbf{v} \tag{1.1.0-14}$$

The total derivative dc/dt We now switch on our outboard motor and race about the river, sometimes upstream, sometimes downstream, or across the current. As we peer over the side of our boat, we measure fish concentration as a function of time while following an arbitrary path across the water:

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + \nabla c \cdot \mathbf{v}_{(b)} \tag{1.1.0-15}$$

Here $\mathbf{v}_{(b)}$ denotes the velocity of the boat.

Exercise 1.1.0-1 Let A be any real scalar field, spatial vector field, or second-order tensor field. Show that[1]

$$\frac{d_{(m)}A}{dt} = \frac{\partial A}{\partial t} + \nabla A \cdot \mathbf{v}$$

Exercise 1.1.0-2 Let $\mathbf{a} = \mathbf{a}(\mathbf{z}, t)$ be some vector field that is a function of position and time.

i) Show that

$$\frac{d_{(m)}\mathbf{a}}{dt} = \left(\frac{\partial a^n}{\partial t} + a^n{}_{,i}\, v^i \right) \mathbf{g}_n$$

ii) Show that

$$\frac{d_{(m)}\mathbf{a}}{dt} = \left(\frac{\partial a_n}{\partial t} + a_{n,i}\, v^i \right) \mathbf{g}^n$$

Exercise 1.1.0-3 Consider the second-order tensor field $\mathbf{T} = \mathbf{T}(\mathbf{z}, t)$.

i) Show that

$$\frac{d_{(m)}\mathbf{T}}{dt} = \left(\frac{\partial T^{ij}}{\partial t} + T^{ij}{}_{,k}\, v^k \right) \mathbf{g}_i \mathbf{g}_j$$

ii) Show that

$$\frac{d_{(m)}\mathbf{T}}{dt} = \left(\frac{\partial T^i{}_j}{\partial t} + T^i{}_{j,k}\, v^k \right) \mathbf{g}_i \mathbf{g}^j$$

Exercise 1.1.0-4 Show that

$$\begin{aligned} \frac{d_{(m)}(\mathbf{a} \cdot \mathbf{b})}{dt} &= \frac{d_{(m)}}{dt}(a^i b_i) \\ &= \frac{d_{(m)}a^i}{dt} b_i + a^i \frac{d_{(m)}b_i}{dt} \end{aligned}$$

[1] Where I write $(\nabla A) \cdot \mathbf{v}$, some authors say instead $\mathbf{v} \cdot (\nabla A)$. When A is a scalar, there is no difference. When A is either a vector or second-order tensor, the change in notation is the result of a different definition for the gradient operation. See Sections A.6.1 and A.8.1.

Exercise 1.1.0-5

i) Starting with the definition for the velocity vector, prove that

$$\mathbf{v} = \frac{d_{(m)}x^i}{dt}\mathbf{g}_i$$

ii) Determine that, with respect to the cylindrical coordinate system defined in Exercise A.4.1-4,

$$\mathbf{v} = \frac{d_{(m)}r}{dt}\mathbf{g}_r + r\frac{d_{(m)}\theta}{dt}\mathbf{g}_\theta + \frac{d_{(m)}z}{dt}\mathbf{g}_z$$

iii) Determine that, with respect to the spherical coordinate system defined in Exercise A.4.1-5,

$$\mathbf{v} = \frac{d_{(m)}r}{dt}\mathbf{g}_r + r\frac{d_{(m)}\theta}{dt}\mathbf{g}_\theta + r\sin\theta\frac{d_{(m)}\varphi}{dt}\mathbf{g}_\varphi$$

Exercise 1.1.0-6 *Path lines* The curve in space along which the material particle ζ travels is referred to as the *path line* for the material particle ζ. The path line may be determined from the motion of the material as described in Section 1.1:

$$\mathbf{z} = \chi(\mathbf{z}_\kappa, t)$$

Here $\mathbf{z}_\kappa$ represents the position of the material particle ζ in the reference configuration $\boldsymbol{\kappa}$; time t is a parameter along the path line that corresponds to any given position $\mathbf{z}_\kappa$.

The path lines may be determined conveniently from the velocity distribution, since velocity is the derivative of position with respect to time following a material particle. The parametric equations of a particle path are the solutions of the differential system

$$\frac{d\mathbf{z}}{dt} = \mathbf{v}$$

or

$$\frac{dz_i}{dt} = v_i \quad \text{for } i = 1, 2, 3$$

The required boundary conditions may be obtained by choosing the reference configuration to be a configuration that the material assumed at some time t_0.

As an example, let the rectangular Cartesian components of $\mathbf{v}$ be

$$v_1 = \frac{z_1}{1+t}, \qquad v_2 = \frac{z_2}{1+2t}, \qquad v_3 = 0$$

and let the reference configuration be that which the material assumed at time $t = 0$. Prove that, in the plane $z_3 = z_{\kappa 3}$, the particle paths or the path lines have the form

$$\frac{z_2}{z_{\kappa 2}} = \left(2\frac{z_1}{z_{\kappa 1}} - 1\right)^{1/2}$$

Exercise 1.1.0-7 *Streamlines* The streamlines for time t form that family of curves to which the velocity field is everywhere tangent at a fixed time t. The parametric equations for the streamlines are solutions of the differential equations

$$\frac{dz_i}{d\alpha} = v_i \quad \text{for } i = 1, 2, 3$$

Here α is a parameter with the units of time, and $d\mathbf{z}/d\alpha$ is tangent to the streamline [see (A.4.1-1)]. Alternatively, we may think of the streamlines as solutions of the differential system

$$\frac{d\mathbf{z}}{d\alpha} \wedge \mathbf{v} = 0$$

or

$$e_{ijk}\frac{dz_j}{d\alpha}v_k = 0 \quad \text{for } i = 1, 2, 3$$

As an example, show that, for the velocity distribution introduced in Exercise 1.1.0-6, the streamlines take the form

$$z_2 = z_{2(0)}\left(\frac{z_1}{z_{1(0)}}\right)^{(1+t)/(1+2t)}$$

for different reference points $(z_{1(0)}, z_{2(0)})$.

Experimentalists sometimes sprinkle particles over a gas–liquid phase interface and take a photograph in which the motion of the particles is not quite stopped (see Figures 3.5.1-1 and 3.5.1-3). The traces left by the particles are proportional to the velocity of the fluid at the surface (so long as we assume that very small particles move with the fluid). For a steady-state flow, such a photograph may be used to construct the particle paths. For an unsteady-state flow, it depicts the streamlines, the family of curves to which the velocity vector field is everywhere tangent.

In two-dimensional flows, the streamlines have a special significance. They are curves along which the stream function (Sections 1.3.7) is a constant. See Exercise 1.3.7-2.

Exercise 1.1.0-8 For the limiting case of steady-state, plane potential flow past a stationary cylinder of radius a with no circulation, the physical components of velocity in cylindrical coordinates are (see Exercise 3.4.2-2)

$$v_r = V\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -V\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

and

$$v_z = 0$$

Show that the family of streamlines is described by

$$\left(1 - \frac{a^2}{r^2}\right) r\sin\theta = C$$

Plot representative members of this family as in Figure 1.1.0-2.

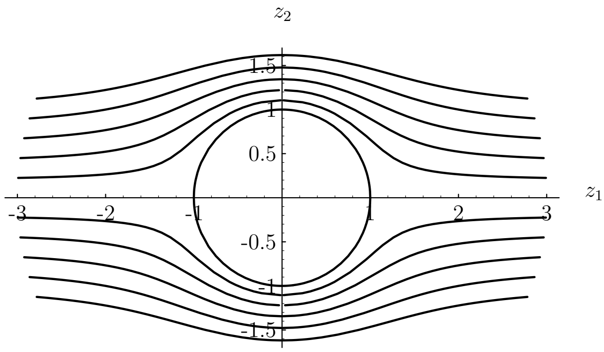

Figure 1.1.0-2. Streamlines for the limiting case of steady-state, plane potential flow past a stationary cylinder with no circulation corresponding to $C = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.$

Exercise 1.1.0-9 *Streak lines* The streak line through the point $\mathbf{z}_{(0)}$ at time t represents the positions at time t of the material particles that at any time $\tau \leq t$ have occupied the place $\mathbf{z}_{(0)}$.

Experimentally we might visualize that smoke, dust, or dye are continuously injected into a fluid at a position $\mathbf{z}_{(0)}$ and that the resulting trails are photographed as functions of time. Each photograph shows a streak line corresponding to the position $\mathbf{z}_{(0)}$ and the time at which the photograph was taken.

We saw in Section 1.1 that the motion χ describes the position $\mathbf{z}$ at time t of the material particle that occupied the position $\mathbf{z}_\kappa$ in the reference configuration:

$$\mathbf{z} = \chi(\mathbf{z}_\kappa, t)$$

In constructing a streak line, we focus our attention on those material particles that were in the place $\mathbf{z}_{(0)}$ at any time $\tau \leq t$:

$$\mathbf{z}_\kappa = \chi^{-1}\left(\mathbf{z}_{(0)}, \tau\right)$$

The parametric equations of the streak line through the point $z_{(0)}$ at time t are obtained by eliminating $\mathbf{z}_\kappa$ between these equations:

$$\mathbf{z} = \chi\left(\chi^{-1}\left(\mathbf{z}_{(0)}, \tau\right), t\right)$$

Time $\tau \leq t$ is the parameter along the streak line.

As an example, show that, for the velocity distribution of Exercise 1.1.0-6, the streak line through $\mathbf{z}_{(0)}$ at time t is specified by

$$z_1 = z_{1(0)}\left(\frac{1+t}{1+\tau}\right)$$

$$z_2 = z_{2(0)}\left(\frac{1+2t}{1+2\tau}\right)^{1/2}$$

$$z_3 = z_{3(0)}$$

A streak line corresponding to $t = 4$ is shown in Figure 1.1.0-3. This figure also presents two of the path lines from Exercise 1.1.0-6 corresponding to $\tau = 0$ and 0.5 that contribute

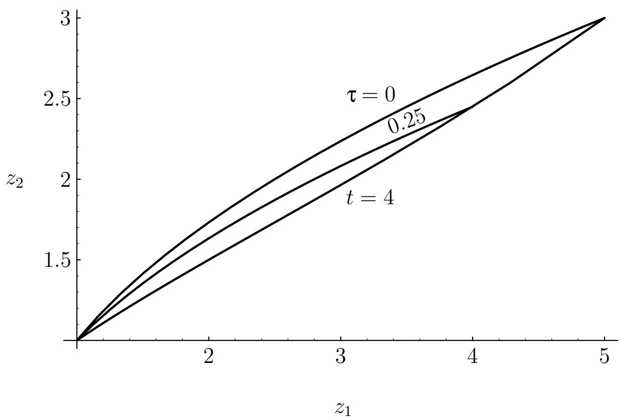

Figure 1.1.0-3. Starting from the top, we see two path lines from Exercise 1.1.0-6) corresponding to $\tau = 0$ and 0.25. The bottom curve is the streak line corresponding to $t = 4$.

to this streak line. The path line corresponding to $\tau = 0$ extends to the right tip of the streak line. It represents the first particle contributing to the streak line. A path line corresponding to $\tau = 4$ would merely be a point at the left tip of the streak line, the origin of the streak line (0, 0). It would represent the particle currently leaving the origin.

Exercise 1.1.0-10 Show that, for a velocity distribution that is independent of time, the path lines, streamlines, and streak lines coincide.

Hint: In considering the path line, take as the boundary condition

$$\text{at } t = \tau : \mathbf{z} = \mathbf{z}_{(0)}$$

This suggests the introduction of a new variable $\alpha \equiv t - \tau$, which denotes time measured since the particle passed through the position $\mathbf{z}_{(0)}$.

1.2 Frame

1.2.1 Changes of Frame

The Chief of the United States Weather Bureau in Milwaukee announces that a tornado was sighted in Chicago at 3 P.M. (Central Standard Time). In Chicago, Harry reports that he saw a black funnel cloud about two hours ago at approximately 800 North and 2400 West. Both men described the same event with respect to their own particular frame of reference.

The time of some occurrence may be specified only with respect to the time of some other event, the *frame of reference for time*. This might be the time at which a stopwatch was started or an electric circuit was closed. The Chief reported the time at which the tornado was sighted relative to the mean time at which the sun appeared overhead on the Greenwich meridian. Harry gave the time relative to his conversation.

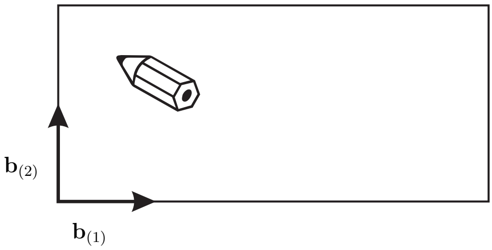

Figure 1.2.1-1. Pencil points away from the direction of $\mathbf{b}_{(1)}$ and toward the direction of $\mathbf{b}_{(2)}$.

A *frame of reference for position* might be the walls of a laboratory, the fixed stars, or the shell of a space capsule that is following an arbitrary trajectory. When the Chief specified Chicago, he meant the city at 41° north and 87° west measured relative to the equator and the Greenwich meridian. Harry thought in terms of eight blocks north of Madison Avenue and 24 blocks west of State Street. More generally, a frame of reference for position is a set of objects whose mutual distances remain unchanged during the period of observation and which do not all lie in the same plane.

To help you get a better physical feel for these ideas, let us consider two more examples.

Extend your right arm and take as your frame of reference for position the direction of your right arm, the direction of your eyes, and the direction of your spine. Stand out at the street with your eyes fixed straight ahead. A car passes in the direction of your right arm. If you were standing facing the street on the opposite side, the automobile would appear to pass in the opposite direction from your right arm.

Lay a pencil on your desk as shown in Figure 1.2.1-1 and take the edges of the desk that meet in the left-hand front corner as your frame of reference for position. The pencil points away from $\mathbf{b}_{(1)}$ and toward $\mathbf{b}_{(2)}$. Without moving the pencil, walk around to the left-hand side and take as your new frame of reference for position the edges of the desk that meet at the left-hand rear corner. The pencil now appears to point toward the intersection of $\mathbf{b}^*_{(1)}$ and $\mathbf{b}^*_{(2)}$ in Figure 1.2.1-2.

Since all of the objects defining a frame of reference do not lie in the same plane, we may visualize replacing them by three mutually orthogonal unit vectors. Let us view a typical point z in this space with respect to two such frames of reference: the $\mathbf{b}_{(i)}$ ($i = 1, 2, 3$) in Figure 1.2.1-3 and the $\mathbf{b}^*_{(j)}$ ($j = 1, 2, 3$) in Figure 1.2.1-4.

An orthogonal transformation preserves both lengths and angles (Section A.5.2). Let $\mathbf{Q}$ be the orthogonal transformation that describes the rotation and (possibly) reflection that takes the $\mathbf{b}_{(i)}$ in Figure 1.2.1-3 into the vectors $\mathbf{Q} \cdot \mathbf{b}_{(i)}$, which are seen in Figure 1.2.1-4 with respect to the starred frame of reference for position. A reflection allows for the possibility that an observer in the new frame looks at the old frame through a mirror. Alternatively, a reflection allows for the possibility that two observers orient themselves oppositely, one choosing to work in terms of a right-handed frame of reference for position and the other in terms of a left-handed one. [For more on this point, I suggest that you read Truesdell (1966a, p. 22) as well as Truesdell and Noll (1965, pp. 24 and 47).]

The vector $\left(\mathbf{z} - \mathbf{z}_{(0)}\right)$ in Figure 1.2.1-3 becomes $\mathbf{Q} \cdot \left(\mathbf{z} - \mathbf{z}_{(0)}\right)$ when viewed in the starred frame shown in Figure 1.2.1-4. From Figure 1.2.1-4, it follows as well that

$$\mathbf{z}^* - \mathbf{z}^*_{(0)} = \mathbf{Q} \cdot \left(\mathbf{z} - \mathbf{z}_{(0)}\right) \tag{1.2.1-1}$$

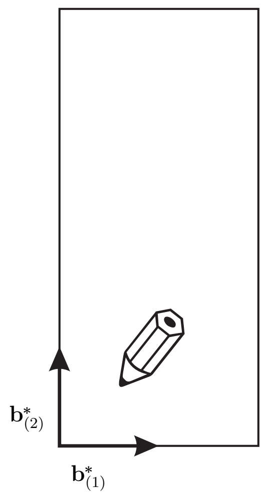

Figure 1.2.1-2. Pencil points toward the direction of $\mathbf{b}^*_{(1)}$ and $\mathbf{b}^*_{(2)}$.

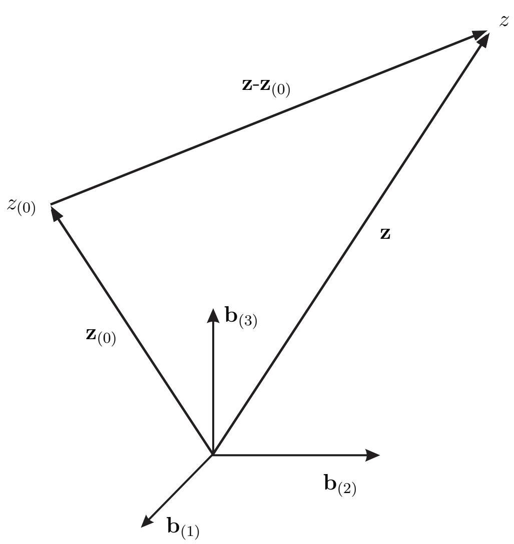

Figure 1.2.1-3. The points z and $z_{(0)}$ are located by the position vectors $\mathbf{z}$ and $\mathbf{z}_{(0)}$ with respect to the frame of reference for position $\left(\mathbf{b}_{(1)}, \mathbf{b}_{(2)}, \mathbf{b}_{(3)}\right)$.

Similarly, $\left(\mathbf{z}^* - \mathbf{z}^*_{(0)}\right)$ in Figure 1.2.1-4 is seen as $\mathbf{Q}^T \cdot \left(\mathbf{z}^* - \mathbf{z}^*_{(0)}\right)$ when observed with respect to the unstarred frame in Figure 1.2.1-5. Figure 1.2.1-5 also makes it clear that

$$\mathbf{z} - \mathbf{z}_{(0)} = \mathbf{Q}^T \cdot \left(\mathbf{z}^* - \mathbf{z}^*_{(0)}\right) \tag{1.2.1-2}$$

Let $\mathbf{z}$ and t denote a position and time in the old frame; $\mathbf{z}^*$ and t^* are the corresponding position and time in the new frame. We can extend the discussion above to conclude that the

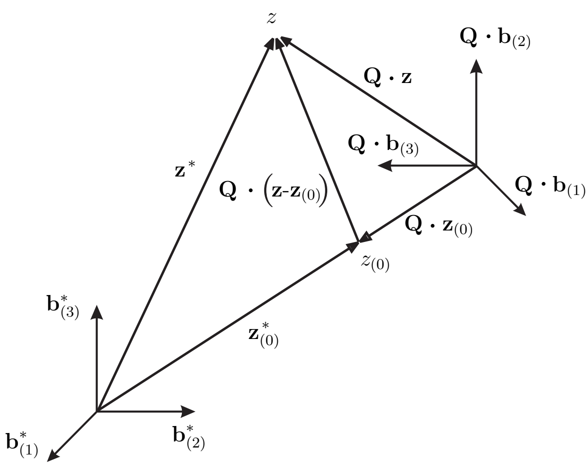

Figure 1.2.1-4. The points z and $z_{(0)}$ are located by the position vectors $\mathbf{z}^*$ and $\mathbf{z}^*_{(0)}$ with respect to the starred frame of reference for position $\left(\mathbf{b}^*_{(1)},\ \mathbf{b}^*_{(2)},\ \mathbf{b}^*_{(3)}\right)$. With respect to the starred frame of reference, the unstarred frame is seen as $\left(\mathbf{Q}\cdot\mathbf{b}_{(1)},\ \mathbf{Q}\cdot\mathbf{b}_{(2)},\ \mathbf{Q}\cdot\mathbf{b}_{(3)}\right)$.

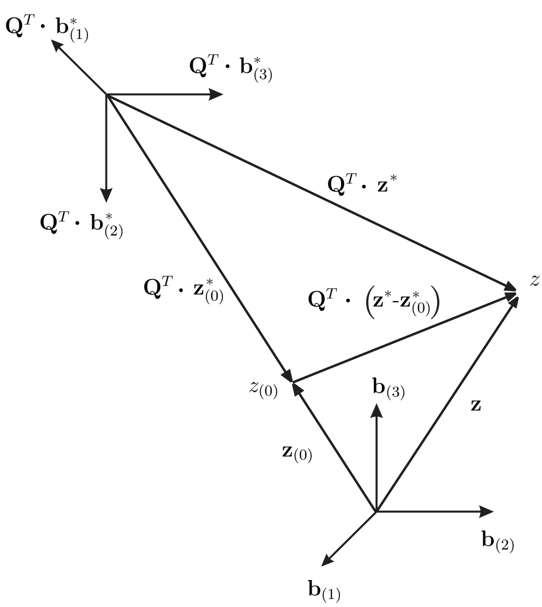

Figure 1.2.1-5. With respect to the unstarred frame of reference, the starred frame is seen as $\left(\mathbf{Q}^T\cdot\mathbf{b}^*_{(1)},\ \mathbf{Q}^T\cdot\mathbf{b}^*_{(2)},\ \mathbf{Q}^T\cdot\mathbf{b}^*_{(3)}\right)$.

most general change of frame is of the form

$$\mathbf{z}^* = \mathbf{z}^*_{(0)}(t) + \mathbf{Q}(t) \cdot \left(\mathbf{z} - \mathbf{z}_{(0)}\right) \tag{1.2.1-3}$$

$$t^* = t - a \tag{1.2.1-4}$$

where we allow the two frames discussed in Figures 1.2.1-3 and 1.2.1-4 to rotate and translate with respect to one another as functions of time. The quantity a is a real number. Equivalently, we could also write

$$\mathbf{z} = \mathbf{z}_{(0)}(t) + \mathbf{Q}^T \cdot \left(\mathbf{z}^* - \mathbf{z}^*_{(0)}\right) \tag{1.2.1-5}$$

$$t = t^* + a \tag{1.2.1-6}$$

It is important to carefully distinguish between a frame of reference for position and a coordinate system. Any coordinate system whatsoever can be used to locate points in space with respect to three vectors defining a frame of reference for position and their intersection, although I recommend that admissible coordinate systems be restricted to those whose axes have a time-invariant orientation with respect to the frame. Let (z_1, z_2, z_3) be a rectangular Cartesian coordinate system associated with the frame of reference $\left(\mathbf{b}_{(1)}, \mathbf{b}_{(2)}, \mathbf{b}_{(3)}\right)$; similarly, let $\left(z_1^*, z_2^*, z_3^*\right)$ be a rectangular Cartesian coordinate system associated with another frame of reference $\left(\mathbf{b}^*_{(1)}, \mathbf{b}^*_{(2)}, \mathbf{b}^*_{(3)}\right)$. We will say that these two coordinate systems are the *same* if the orientation of the basis fields $\mathbf{e}_i$ with respect to the vectors $\mathbf{b}_{(j)}$ is identical to the orientation of the basis fields $\mathbf{e}^*_i$ with respect to the vectors $\mathbf{b}^*_{(j)}$:

$$\mathbf{e}_i \cdot \mathbf{b}_{(j)} = \mathbf{e}^*_i \cdot \mathbf{b}^*_{(j)} \quad \text{for all } i, j = 1, 2, 3 \tag{1.2.1-7}$$

We will generally find it convenient to use the same coordinate system in discussing two different frames of reference.

Let us use the *same* rectangular Cartesian coordinate system to discuss the change of frame illustrated in Figures 1.2.1-4 and 1.2.1-5. The orthogonal tensor

$$\mathbf{Q} = Q_{ij}\mathbf{e}^*_i\mathbf{e}_j \tag{1.2.1-8}$$

describes the rotation (and possibly reflection) that transforms the basis vectors $\mathbf{e}_j$ $(j = 1, 2, 3)$ into the vectors

$$\mathbf{Q} \cdot \mathbf{e}_j = Q_{ij}\mathbf{e}^*_i \tag{1.2.1-9}$$

which are vectors expressed in terms of the starred frame of reference for position. The rectangular Cartesian components of $\mathbf{Q}$ are defined by the angles between the $\mathbf{e}^*_i$ and the $\mathbf{Q} \cdot \mathbf{e}_j$:

$$Q_{ij} = \mathbf{e}^*_i \cdot \left(\mathbf{Q} \cdot \mathbf{e}_j\right) \tag{1.2.1-10}$$

The vector $\left(\mathbf{z} - \mathbf{z}_{(0)}\right)$ in Figure 1.2.1-3 becomes

$$\mathbf{Q} \cdot \left(\mathbf{z} - \mathbf{z}_{(0)}\right) = Q_{ij}\left(z_j - z_{(0)j}\right)\mathbf{e}^*_i \tag{1.2.1-11}$$

when viewed in the starred frame shown in Figure 1.2.1-4. From Figure 1.2.1-4, it follows as well that

$$z^*_i\mathbf{e}^*_i = z^*_{(0)i}\mathbf{e}^*_i + Q_{ij}\left(z_j - z_{(0)j}\right)\mathbf{e}^*_i \tag{1.2.1-12}$$

We speak of a quantity as being *frame indifferent* if it remains unchanged or invariant under all changes of frame. A *frame-indifferent scalar b* does not change its value:

$$b^* = b \tag{1.2.1-13}$$

A *frame-indifferent spatial vector* remains the same directed line element under a change of frame in the sense that if

$$\mathbf{u} = \mathbf{z}_1 - \mathbf{z}_2$$

then

$$\mathbf{u}^* = \mathbf{z}_1^* - \mathbf{z}_2^*$$

From (1.2.1-3),

$$\begin{aligned} \mathbf{u}^* &= \mathbf{Q} \cdot (\mathbf{z}_1 - \mathbf{z}_2) \\ &= \mathbf{Q} \cdot \mathbf{u} \end{aligned} \tag{1.2.1-14}$$

A *frame-indifferent second-order tensor* is one that transforms frame-indifferent spatial vectors into frame-indifferent spatial vectors. If

$$\mathbf{u} = \mathbf{T} \cdot \mathbf{w} \tag{1.2.1-15}$$

the requirement that **T** be a frame-indifferent second-order tensor is

$$\mathbf{u}^* = \mathbf{T}^* \cdot \mathbf{w}^* \tag{1.2.1-16}$$

where

$$\begin{aligned} \mathbf{u}^* &= \mathbf{Q} \cdot \mathbf{u} \\ \mathbf{w}^* &= \mathbf{Q} \cdot \mathbf{w} \end{aligned} \tag{1.2.1-17}$$

This means that

$$\begin{aligned} \mathbf{Q} \cdot \mathbf{u} &= \mathbf{T}^* \cdot \mathbf{Q} \cdot \mathbf{w} \\ &= \mathbf{Q} \cdot \mathbf{T} \cdot \mathbf{w} \end{aligned} \tag{1.2.1-18}$$

which implies

$$\mathbf{T} = \mathbf{Q}^T \cdot \mathbf{T}^* \cdot \mathbf{Q} \tag{1.2.1-19}$$

or

$$\mathbf{T}^* = \mathbf{Q} \cdot \mathbf{T} \cdot \mathbf{Q}^T \tag{1.2.1-20}$$

The importance of changes of frame will become apparent in Section 2.3.1, where the principle of frame indifference is introduced. This principle will be used repeatedly in discussing representations for material behavior and in preparing empirical data correlations.

The material in this section is drawn from Truesdell and Toupin (1960, p. 437), Truesdell and Noll (1965, p. 41), and Truesdell (1966a, p. 22).

Exercise 1.2.1-1 Let T be a frame-indifferent scalar field. Starting with the definition of the gradient of a scalar field in Section A.3.1, show that the gradient of T is frame indifferent:

$$\nabla T^{*} \equiv (\nabla T)^{*} = \mathbf{Q} \cdot \nabla T$$

Exercise 1.2.1-2 In order that $\boldsymbol{\epsilon}$ (defined in Exercise A.7.2-11) be a frame-indifferent third-order tensor field, prove that

$$\boldsymbol{\epsilon}^{*} = (\det \mathbf{Q}) e_{ijk} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k$$

1.2.2 Equivalent Motions

In Section 1.1, I described the motion of a material with respect to some frame of reference by

$$\mathbf{z} = \chi(\mathbf{z}_{\kappa}, t) \tag{1.2.2-1}$$

where we understand that the form of this relation depends upon the choice of reference configuration $\boldsymbol{\kappa}$. According to our discussion in Section 1.2.1, the same motion with respect to some new frame of reference is represented by

$$\begin{aligned}\mathbf{z}^{*} &= \chi^{*}(\mathbf{z}_{\kappa}^{*}, t^{*}) \\ &= \mathbf{z}_0^{*}(t) + \mathbf{Q}(t) \cdot [\chi(\mathbf{z}_{\kappa}, t) - \mathbf{z}_0]\end{aligned} \tag{1.2.2-2}$$

We will say that any two motions χ and χ^{*} related by an equation of the form of (1.2.2-2) are *equivalent motions*.

Let us write (1.2.2-2) in an abbreviated form:

$$\mathbf{z}^{*} = \mathbf{z}_0^{*} + \mathbf{Q} \cdot (\mathbf{z} - \mathbf{z}_0) \tag{1.2.2-3}$$

The material derivative of this equation gives

$$\mathbf{v}^{*} = \frac{d\mathbf{z}_0^{*}}{dt} + \frac{d\mathbf{Q}}{dt} \cdot (\mathbf{z} - \mathbf{z}_0) + \mathbf{Q} \cdot \mathbf{v} \tag{1.2.2-4}$$

or

$$\mathbf{v}^{*} - \mathbf{Q} \cdot \mathbf{v} = \frac{d\mathbf{z}_0^{*}}{dt} + \frac{d\mathbf{Q}}{dt} \cdot (\mathbf{z} - \mathbf{z}_0) \tag{1.2.2-5}$$

In view of (1.2.2-3), we may write

$$\begin{aligned}\mathbf{z} - \mathbf{z}_0 &= \mathbf{Q}^{T} \cdot \mathbf{Q} \cdot (\mathbf{z} - \mathbf{z}_0) \\ &= \mathbf{Q}^{T} \cdot \left(\mathbf{z}^{*} - \mathbf{z}_0^{*}\right)\end{aligned} \tag{1.2.2-6}$$

This allows us to express (1.2.2-5) as

$$\begin{aligned}\mathbf{v}^{*} - \mathbf{Q} \cdot \mathbf{v} &= \frac{d\mathbf{z}_0^{*}}{dt} + \left(\frac{d\mathbf{Q}}{dt} \cdot \mathbf{Q}^{T}\right) \cdot \left(\mathbf{z}^{*} - \mathbf{z}_0^{*}\right) \\ &= \frac{d\mathbf{z}_0^{*}}{dt} + \mathbf{A} \cdot \left(\mathbf{z}^{*} - \mathbf{z}_0^{*}\right)\end{aligned} \tag{1.2.2-7}$$

where

$$\mathbf{A} \equiv \frac{d\mathbf{Q}}{dt} \cdot \mathbf{Q}^T \tag{1.2.2-8}$$

We refer to the second-order tensor **A** as the *angular velocity tensor of the starred frame with respect to the unstarred frame* (Truesdell 1966a, p. 24).

Since **Q** is an orthogonal tensor,

$$\mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{I}^* \tag{1.2.2-9}$$

Taking the material derivative of this equation, we have

$$\begin{aligned}\mathbf{A} = \frac{d\mathbf{Q}}{dt} \cdot \mathbf{Q}^T &= -\mathbf{Q} \cdot \frac{d\mathbf{Q}^T}{dt} \\ &= -\mathbf{Q} \cdot \left(\frac{d\mathbf{Q}}{dt}\right)^T \\ &= -\mathbf{A}^T\end{aligned} \tag{1.2.2-10}$$

In this way we see that the angular velocity tensor is skew symmetric.

The *angular velocity vector of the unstarred frame with respect to the starred frame* $\boldsymbol{\omega}$ is defined as

$$\boldsymbol{\omega} \equiv \frac{1}{2}\text{tr}(\boldsymbol{\epsilon}^* \cdot \mathbf{A}) \tag{1.2.2-11}$$

The third-order tensor $\boldsymbol{\epsilon}$ is introduced in Exercises A.7.2-11 and A.7.2-12 (see also Exercise 1.2.1-2), where tr denotes the trace operation defined in Section A.7.3. Let us consider the following spatial vector in rectangular Cartesian coordinates:

$$\begin{aligned}\boldsymbol{\omega} \wedge \left(\mathbf{z}^* - \mathbf{z}_0^*\right) &= \text{tr}\left(\boldsymbol{\epsilon}^* \cdot \left[\left(\mathbf{z}^* - \mathbf{z}_0^*\right)\boldsymbol{\omega}\right]\right) \\ &= \text{tr}\left(\boldsymbol{\epsilon}^* \cdot \left\{\left[\mathbf{z}^* - \mathbf{z}_0^*\right]\left[\frac{1}{2}\text{tr}(\boldsymbol{\epsilon}^* \cdot \mathbf{A})\right]\right\}\right) \\ &= e_{ijk}\left(z_k^* - z_{0\,k}^*\right)\left(\frac{1}{2}e_{jmn}A_{nm}\right)\mathbf{e}_i^* \\ &= \frac{1}{2}\left(z_k^* - z_{0\,k}^*\right)(A_{ij} - A_{ki})\mathbf{e}_i^* \\ &= \left(z_k^* - z_{0\,k}^*\right)A_{ik}\mathbf{e}_i^* \\ &= \mathbf{A} \cdot \left(\mathbf{z}^* - \mathbf{z}_0^*\right)\end{aligned} \tag{1.2.2-12}$$

We may consequently write (1.2.2-7) in terms of the angular velocity of the unstarred frame with respect to the starred frame (Truesdell and Toupin 1960, p. 437):

$$\mathbf{v}^* = \frac{d\mathbf{z}_0^*}{dt} + \boldsymbol{\omega} \wedge [\mathbf{Q} \cdot (\mathbf{z} - \mathbf{z}_0)] + \mathbf{Q} \cdot \mathbf{v} \tag{1.2.2-13}$$

The material in this section is drawn from Truesdell and Noll (1965, p. 42) and Truesdell (1966a, p. 22).

Exercise 1.2.2-1

i) Show that velocity is not frame indifferent.
ii) Show that at any position in euclidean point space a difference in velocities with respect to the same frame is frame indifferent.

Exercise 1.2.2-2 *Acceleration*

i) Determine that (Truesdell 1966a, p. 24)

$$\frac{d_{(m)}\mathbf{v}^*}{dt} = \frac{d^2\mathbf{z}_0^*}{dt^2} + 2\mathbf{A}\cdot\left(\mathbf{v}^* - \frac{d\mathbf{z}_0^*}{dt}\right) + \left(\frac{d\mathbf{A}}{dt} - \mathbf{A}\cdot\mathbf{A}\right)\cdot\left(\mathbf{z}^* - \mathbf{z}_0^*\right) + \mathbf{Q}\cdot\frac{d_{(m)}\mathbf{v}}{dt}$$

ii) Prove that (Truesdell and Toupin 1960, p. 440)

$$\frac{d_{(m)}\mathbf{v}^*}{dt} = \frac{d^2\mathbf{z}_0^*}{dt^2} + \left(\frac{d_{(m)}\mathbf{A}}{dt} + \mathbf{A}\cdot\mathbf{A}\right)\cdot\mathbf{Q}\cdot(\mathbf{z} - \mathbf{z}_0) + 2\mathbf{A}\cdot\mathbf{Q}\cdot\mathbf{v} + \mathbf{Q}\cdot\frac{d_{(m)}\mathbf{v}}{dt}$$

iii) Prove that

$$\boldsymbol{\omega}\wedge\left[\boldsymbol{\omega}\wedge\left(\mathbf{z}^* - \mathbf{z}_0^*\right)\right] = \mathbf{A}\cdot\mathbf{A}\cdot\left(\mathbf{z}^* - \mathbf{z}_0^*\right)$$

$$\frac{d\boldsymbol{\omega}}{dt}\wedge\ \wedge\left(\mathbf{z}^* - \mathbf{z}_0^*\right) = \frac{d\mathbf{A}}{dt}\cdot\left(\mathbf{z}^* - \mathbf{z}_0^*\right)$$

and

$$\boldsymbol{\omega}\wedge(\mathbf{Q}\cdot\mathbf{v}) = \mathbf{A}\cdot\mathbf{Q}\cdot\mathbf{v}$$

iv) Conclude that (Truesdell and Toupin 1960, p. 438)

$$\frac{d_{(m)}\mathbf{v}^*}{dt} = \frac{d^2\mathbf{z}_0^*}{dt^2} + \frac{d\boldsymbol{\omega}}{dt}\wedge\ \cdot[\mathbf{Q}\cdot(\mathbf{z} - \mathbf{z}_0)] + \boldsymbol{\omega}\wedge\{\boldsymbol{\omega}\wedge[\mathbf{Q}\cdot(\mathbf{z} - \mathbf{z}_0)]\} + 2\boldsymbol{\omega}\ \wedge\ \cdot(\mathbf{Q}\cdot\mathbf{v}) + \mathbf{Q}\cdot\frac{d_{(m)}\mathbf{v}}{dt}$$

Exercise 1.2.2-3 Give an example of a scalar that is not frame indifferent.

Hint: What vector is not frame indifferent?

Exercise 1.2.2-4 *Motion of a rigid body* Determine that the velocity distribution in a rigid body may be expressed as

$$\mathbf{v}^* = \frac{d\mathbf{z}_0^*}{dt} + \boldsymbol{\omega} \wedge \left(\mathbf{z}^* - \mathbf{z}_0^*\right)$$

What is the relation of the unstarred frame to the body in this case?

1.3 Mass

1.3.1 Conservation of Mass

This discussion of mechanics is based upon several postulates. The first is

Conservation of mass The mass of a body is independent of time.

Physically, this means that, if we follow a portion of a material body through any number of translations, rotations, and deformations, the mass associated with it will not vary as a function of time. If ρ is the *mass density* of the body, the mass may be represented as

$$\int_{R_{(m)}} \rho \, dV$$

Here dV denotes that a volume integration is to be performed over the region $R_{(m)}$ of space occupied by the body in its current configuration; in general $R_{(m)}$, or the limits on this integration, is a function of time. The postulate of conservation of mass says that

$$\frac{d}{dt} \int_{R_{(m)}} \rho \, dV = 0 \tag{1.3.1-1}$$

Notice that, like the material particle introduced in Section 1.1, *mass* is a primitive concept. Rather than defining mass, we describe its properties. We have just examined its most important property: It is conserved. In addition, I will require that

$$\rho > 0 \tag{1.3.1-2}$$

and that the mass density be a frame-indifferent scalar,

$$\rho^* = \rho \tag{1.3.1-3}$$

Our next objective will be to determine a relationship that expresses the idea of conservation of mass at each point in a material. To do this, we will find it necessary to interchange the operations of differentiation and integration in (1.3.1-1). Yet the limits on this integral describe the boundaries of the body in its current configuration and generally are functions of time. The next section explores this problem in more detail.

1.3.2 Transport Theorem

Let us consider the operation

$$\frac{d}{dt} \int_{R_{(m)}} \Psi \, dV$$