1

Hybrid Estimation

1.1 Introduction

Common problems in design require that an engineer devise a control or decision algorithm that converts measurements of system and environmental variables into signals that aid in system regulation. For example, a control node converts sensor outputs into an actuating signal that moves the system toward the desired operating point and keeps it there. At this foundational level, the engineer must formulate a mapping from the system observables into an action or report; for example, a feedback regulator converts the measured outputs of the system to be controlled (the *plant*) into an input that stabilizes the system.

Design is made difficult by disturbances internal to the system and by noise at its output. For example, there may be no sensors that measure those plant variables most useful for regulation, or, if measured, the variables may be masked by noise in the sensor-to-regulator link. Lacking omniscience, an engineer must process the available measurements to produce a good approximation to relevant but "hidden" variables. And this inference must be done on-line. The processing algorithm must not only be adapted to the incoming data stream, it must be of a form that can be implemented: An implementable estimation algorithm is an explicit mapping of the sensor output process (the *measurements*) into a (nearly) concurrent estimate of the required variables. In the applications studied here, the need for contemporaneous response limits consideration to finite-dimensional recursive algorithms; new observations are integrated into an estimate in an accretive manner.

Analytical design in estimation and control begins with a formal mathematical description of the system to be controlled (the *plant model*). The model delineates the response of the plant to endogenous actuating signals as well as representing the influence of exogenous disturbances common to the application. The system designer selects a control policy or a state estimation algorithm based in large part upon the behaviors predicted by the model. The practicality of analytic procedures is linked closely to the realism of the plant model. However, realism must be tempered by the need to have a model that is simultaneously flexible and tractable.

One useful paradigm phrases the plant model in terms of a set of nonlinear stochastic differential equations. Let us start with a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a time interval of interest, $[0, \mathsf{T}]$. On this space there is a right-continuous filtration $\{\mathcal{F}_t; 0 \leq t \leq \mathsf{T}\}$ and right-continuous, $\mathcal{F}_t$-adapted random processes, $\{\Phi_t\}$, $\{w_t\}$, and $\{n_t\}$. Subject to initial conditions χ_0 and g_0, the plant model is written:

plant model

$$d\chi_t = \mathbf{f}(\chi_t, \upsilon_t, \Phi_t)\, dt + \mathbf{g}(\chi_t, \upsilon_t, \Phi_t)\, dw_t, \tag{1.1}$$

$$dg_t = \mathbf{r}(\chi_t, \upsilon_t, \Phi_t)\, dt + \mathbf{s}(\chi_t, \upsilon_t, \Phi_t)\, dn_t, \tag{1.2}$$

where $\{\upsilon_t\}$ is an s-dimensional actuating process (the *plant input*), $\{g_t\}$ is an r-dimensional observation process (the *plant output*), and $\{\chi_t\}$ is an n-dimensional internal process (the *plant state*). Equation (1.1) describes the temporal evolution of the internal variables within the plant, and (1.2) describes the sensor outputs available for estimation and/or control.

This plant model is more complicated than that encountered in introductory studies of feedback control. In applications, even when the actuating process is specified, the realizations of the state and output paths are unpredictable – there are many effects not well captured in a deterministic model. Chance influences in the plant and sensor are represented by the stochastic processes in (1.1) and (1.2). Various accretive effects are represented by $\{w_t\}$ and $\{n_t\}$; for example, $\{w_t\}$ could describe the high frequency modes ignored in a low-dimensional plant model, and $\{n_t\}$ could describe noise at the sensor output. The *environmental process*, $\{\Phi_t\}$, denotes external conditions of a more global sort that affect plant operation. The value of $\{\Phi_t\}$ might indicate the operational status of a subelement within the plant, external conditions that influence the plant dynamics (e.g., temperature), the level of loads placed upon the system by linked elements, etc. In contrast to $\{w_t\}$ and $\{n_t\}$, which tend to be aggregations of small increments, Φ_t may symbolize temporally distinct events. (Friedland called Φ_t the *metastate* when used in the context of adaptive control; see [Fri96, Chapter 10].) All of these disturbance processes are viewed by the designer as exogenous.

In both estimation and the control, the output signal, $\{g_t\}$, is processed to create causal estimates of important system variables. A *filter* provides estimates of the current values of both the plant state vector and the environmental process. A *predictor* estimates future values of the same variables. Often, the environmental process has a character fundamentally different from the plant state. The value of Φ_t

may be a symbolic variable (e.g., $\Phi_t \in \{$*normal operation, degraded operation*$\}$). In this event, the average value of Φ_t has no meaning. Rather, the probability distribution of Φ_t is required to properly assess the status of the plant. Denote the filtration generated by $\{g_t\}$ by $\{\mathcal{G}_t\}$. If mean square error is used as a performance index, the estimation problem can be posed as follows:

> Find an explicit processing algorithm to generate (or approximate) the mean plant state $\hat{\chi}_t = E[\chi_t \mid \mathcal{G}_t]$ and the $\mathcal{G}_t$-probability distribution of Φ_t.

There are applications in which even this will not suffice and more comprehensive statistical properties of the plant processes are required.

Unfortunately, even when formal descriptions of the exogenous processes are integrated with (1.1) and (1.2), an elementary solution to this estimation problem does not currently exist. There is, however, one special case in which astounding success has been achieved. So much so that the solution thus derived is used in circumstances far removed from those in which it was developed. Specifically, suppose that the system has "smooth" nonlinearities, that the plant noise, $\{w_t\}$, is a Brownian motion, and that the environmental process, $\{\Phi_t\}$, is constant with known value Φ_c. Associated with Φ_c there is a nominal operating condition, both in the state and in the actuating signal labeled (χ_n, υ_n). Frequently (χ_n, υ_n) is a condition of plant stasis: $f(\chi_n, \upsilon_n, \Phi_c) = 0$. The operating condition (or *regime*) is known by different names: in the process control industry, (χ_n, υ_n) is referred to as the set point or the operating point; in aircraft flight control, (χ_n, υ_n) is referred to as the trim condition; in other applications, (χ_n, υ_n) is simply the reference point. We will use these terms interchangeably and note in this context that Φ_t simply points to the operating mode or regime with its value having no intrinsic meaning.

For a particular regime, there is a local description of the plant phrased in terms of a set of perturbation variables. These are defined as the (usually small) deviations in state and excitation from the set point: $x_t = \chi_t - \chi_n$; $u_t = \upsilon_t - \upsilon_n$. Using orthodox methods and neglecting higher order terms, the perturbation processes are commonly represented by a linear stochastic differential equation with initial condition taken to be Gaussian: x_0 is $\mathbf{N}(\hat{x}_0, P_{xx}(0))$, and

$$dx_t = (Ax_t + Bu_t)\,dt + C\,dw_t, \tag{1.3}$$

where $\{w_t\}$ is a Brownian motion with intensity $W(d\langle w, w\rangle_t = W dt)$. Call $\{x_t\}$ the *base-state process* to distinguish it from the plant state process, $\{\chi_t\}$; call $\{u_t\}$ the *regulation signal* to distinguish it from the plant input, $\{\upsilon_t\}$. Equation (1.3) relates the base-state to the inputs $\{u_t\}$ (endogenous) and $\{w_t\}$ (exogenous). The base-state excitation is a Brownian motion with intensity $CWC' = R_\chi$. Of course, if the plant is linear over a large region of the state space, (1.3) is valid without consideration of the set point. In such applications, it is understood that χ_n and υ_n are both zero.

The set point is known ($\mathcal{P}[\Phi_t \equiv \Phi_c] = 1$) and need not be estimated, but the plant state is frequently not known and must be inferred from sensor outputs. Suppose a sensor provides a noisy but linear plant state measurement,

plant state measurement: time-continuous

$$dy_t = H\chi_t\,dt + dn_t, \tag{1.4}$$

where $\{n_t\}$ is a Brownian motion independent of $\{w_t\}$, with intensity $R_x > 0$ ($d\langle n, n\rangle_t = R_x\,dt$), and $y_0 = 0$. By subtracting the contribution of the set point from the output, (1.4) can be written as a noisy, linear measurement of the base-state: $dy_t - H\chi_n dt = Hx_t\,dt + dn_t$. The innovation increment $d\nu_t = dy_t - d\hat{y}_t$ can be written $H\tilde{x}_t\,dt + dn_t$, where $\tilde{x}_t = x_t - \hat{x}_t$. When there is only one sensor, $g_t \equiv y_t$. To differentiate this case from others that follow, denote the filtration generated by $\{y_t\}$ by $\{\mathcal{Y}_t\}$ ($= \mathcal{G}_t$ in this case), where a circumflex may be used to denote $\mathcal{Y}_t$-expectation if no confusion will result. Equations (1.3) and (1.4) will be called a *linear–Gauss–Markov* (LGM) model even when x_0 is not Gaussian. Although the observation is unconventional, the regime offset is known and is accommodated in a direct fashion. The base-state estimator is known for the LGM problem: the Kalman filter. The Kalman filter generates $\{\hat{x}_t\}$ using a simple recursive algorithm. The plant state estimator is $\hat{\chi}_t = \chi_n + \hat{x}_t$.

In the systems we will study, $\{\Phi_t\}$ is not nearly so obliging. Instead of a single operating point, $\{\Phi_t\}$ may move about in its range space in response to the macroevents that influence the plant. The temporal structure of the regime process has a fundamental impact on system analysis. If, for example, $\{\Phi_t\}$ has sample paths that are well described by a diffusion process, then $\{\Phi_t\}$ can be integrated into (1.1) as an additional plant state. This is an attractive option when the time constants of $\{\Phi_t\}$ are comparable with those of the plant, though this inclusion compounds the plant nonlinearity.

In other applications, $\{\Phi_t\}$ has a distinguishing feature that precludes orthodox state augmentation. Suppose the plant has S possible operating regimes, and at any particular time, Φ_t takes on a value selected from a set of size S: $\Phi_t \in \{\Phi_i; i \in \mathbf{S}\}$. The plant now has S possible reference points (or set points, etc.), and these are identified with the S possible values of $\{\Phi_t\}$; that is, there are S vector pairs, $\{(\chi_i, \upsilon_i); i \in \mathbf{S}\}$, which designate the S relevant stasis conditions for the plant. For example, the kth nominal operating point for the plant is (χ_k, υ_k), and if $\Phi_t = \Phi_k$ the plant input and state should be near (χ_k, υ_k).

For simplicity, array the nominal states (respectively nominal actuating signals) as an $n \times S$ matrix $\boldsymbol{\chi}$ (respectively an $s \times \mathrm{S}$ matrix $\boldsymbol{\upsilon}$): $\boldsymbol{\chi} = [\chi_i]$ (respectively

$\boldsymbol{\upsilon} = [\upsilon_i]$). During operation, the system will operate in one regime for a time ($\Phi_t = \Phi_i$ for $t \in [a, b)$) and then suddenly shift ($\Phi_b = \Phi_j$) to another in response to an external event or change in the surrounding environment. In most applications, the discontinuous sample paths of $\{\Phi_t\}$ are an approximation to the continuous though abrupt modal transitions that actually occur. Nevertheless, the representation of $\{\Phi_t\}$ with a process of piecewise constant paths is a useful abstraction when the interval over which the modal transition takes place is short as compared to the important time constants of the plant.

Since the environmental process has a finite state space, $\{\Phi_t\}$ can be represented using a more illuminating notation. Let ϕ_t be a pointer to the current regime: The state space of ϕ_t consists of the S canonical unit vectors in $\mathbb{R}^S$ ($\phi_t \in \{\mathbf{e}_1, \ldots, \mathbf{e}_S\}$). The component in ϕ_t with value one marks the current mode of operation: If $\Phi_t = \Phi_k$ then $\phi_t = \mathbf{e}_k$. The $\{\phi_t\}$ process is called the *modal-state process* to differentiate it from the base-state process. The base-state variables are deviations from the current set point: $x_t = \chi_t - \boldsymbol{\chi}\phi_t$; $u_t = \upsilon_t - \boldsymbol{\upsilon}\phi_t$. The comprehensive state of the system is the composition of the base- and modal-states: The *zygostate* is the pair (x_t, ϕ_t). Since ϕ_t is an indicator vector, the expectation of the modal-state is actually the conditional probability vector $\hat{\phi}_t = [\mathcal{P}\{\phi_i = \mathbf{e}_i \,|\, \mathcal{G}_t\}]$.

Control in a multiregime environment presents some subtle challenges. When the regime is known and constant (e.g., $\phi_t \equiv \mathbf{e}_i$), the actuating signal has a natural decomposition ($\upsilon_t = u_t + \boldsymbol{\upsilon}\phi_t$) into a feedforward component associated with the set point ($\boldsymbol{\upsilon}\mathbf{e}_i = \upsilon_i$) and a feedback component (u_t) that maintains the plant state near the set point ($\chi_t \approx \chi_i$). When the modal-state is neither known nor measured, this implementation is not possible because proper feedforward control cannot be generated. In applications, a variety of replacements for $\{\boldsymbol{\upsilon}\phi_t\}$ have been proposed. We will not explore issues of feedforward control in any depth here. We will simply employ $\{\boldsymbol{\upsilon}\hat{\phi}_t\}$ as the "feedforward" component of the actuating signal: Ideal set point actuation will be replaced with its expectation. Note, however, that a failure to generate the proper feedforward actuating signal has an influence that must be included in the base-state dynamics.

A comprehensive plant model requires a representation of evolution, both intramodal and intermodal. Consider the former first. During an extended (known) modal sojourn, proper control will place and maintain the plant state vector near the correct set point. The natural plant model in this circumstance would be that local model, selected from a family of regime-specific, linear models, associated with the present mode of operation. The modal-state is a pointer, and the intrasojourn model can be written:

$$dx_t = \sum_i ((A_i x_t + B_i(u_t + \boldsymbol{\upsilon}(\hat{\phi}_t - \mathbf{e}_i)))\,dt + C_i\,dw_t)\phi_i, \tag{1.5}$$

where $\{A_i, B_i, C_i; i \in \mathbf{S}\}$ are determined from (1.1) in precisely the way (1.3) was in the unimodal (or unimorphic†) case.

Suppose the plant is in the ith mode ($\phi_t = \mathbf{e}_i$) and the modal estimate is a good one ($\hat{\phi}_t \approx \mathbf{e}_i$). The base-state dynamic equation is the ith selection from the family of models: $(A_i x_t + B_i u_t)\,dt + C_i\,dw_t$. The exogenous excitation is a Brownian motion with intensity $R_\chi(i) = C_i W C_i'$. There is an atypical term in (1.5) that is connected with failure to implement the proper feedforward excitation $(-B_i \upsilon \tilde{\phi}_t \phi_i\,dt)$. When the estimate of ϕ_t is good, this last term is negligible, and the intramodal dynamics are LGM.

The intramodal representation is but a part of the model of plant evolution. When the regime changes, many things can happen to the plant state. There will be no attempt to be exhaustive in this list, but we will encounter situations in which the plant state translates, rotates, and/or is scaled. More specifically, suppose $\{\Phi_t\}$ makes the transition $\mathbf{e}_i \mapsto \mathbf{e}_l$ at time t. Then $\{\chi_t\}$ may experience:

Translation: $\Delta\chi_t = \rho(i,l);\ i \neq l$.
Rotation and/or scaling: $\Delta\chi_t = M(i,l)\chi_{t-};\ i \neq l$,
where $\Delta\chi_t = \chi_t - \chi_{t-}$.

When the mode changes, the plant state may be transformed in a way that creates a path discontinuity. This abrupt change in plant state is an approximation in most cases. But, if the interval over which a change takes place is small, a discontinuous path model may provide a far simpler representation of the state variation than would a continuous path model created from an intricate diffusion process. To fill out the list of transformation matrices, let $\rho(i,i) = 0$, $M(i,i) = 0$; $i \in \mathbf{S}$. The indicator vector of the discontinuity event $\mathbf{e}_i \mapsto \mathbf{e}_l$ at time t can be written as $\phi_i \mathbf{e}_l' \Delta\phi_t$. The plant state discontinuity can be written explicitly as

$$\Delta\chi_t = \sum_{i,l} (M(i,l)\chi_{t-} + \rho(i,l))\phi_i \mathbf{e}_l' \Delta\phi_t.$$

Discontinuities in $\{\chi_t\}$ are reflected directly in $\{x_t\}$, but there is an additional source of base-state discontinuity. When the mode changes $\mathbf{e}_i \mapsto \mathbf{e}_l$, the base-state reference level changes from χ_i to χ_l. Even if the plant state were continuous, the base-state would experience a discontinuity:

$$\Delta x_t = -\boldsymbol{\chi}\Delta\phi_t.$$

These intermodal transition conditions can be combined to yield the base-state

† We say a system may have one or several modes or, equivalently, forms. Hence a single-mode plant is called unimodal (or unimorphic) to distinguish it from a polymodal (polymorphic) system.

discontinuity model:

$$\Delta x_t = \sum_{i,l} (M(i,l)x_{t-} + (\chi_i - \chi_l) + M(i,l)\chi_i + \rho(i,l))\phi_i \mathbf{e}_l' \Delta\phi_t. \quad (1.6)$$

Now combine the intermodal discontinuity with the intramodal dynamics to yield:

base-state model

$$dx_t = \sum_i ((A_i x_t + B_i(u_t - \upsilon\phi_t))\, dt + C_i\, dw_t)\phi_i + \sum_{i,l} (M(i,l)x_t + (\chi_i - \chi_l) + M(i,l)\chi_i + \rho(i,l))\phi_i \mathbf{e}_l' \Delta\phi_t. \quad (1.7)$$

Equation (1.7) is the fundamental model of time-continuous base-state evolution. Its appearance is formidable. Be assured that while the various discontinuity and set point conditions will appear in what follows, in no application will all occur simultaneously! In many cases, (1.7) takes on a strikingly simpler form. It is advantageous to set apart some special instances of (1.7) because they are easier to interpret.

LJS: The most often studied specialization of (1.7) is called a linear jump system (LJS). In an LJS there is no regime-specific set point reference ($\chi = 0$, $\upsilon = 0$), nor are there plant state discontinuities at modal transition [Mar90]. The LJS model is simply

$$dx_t = \sum_i ((A_i x_t + B_i u_t)\, dt + C_i\, dw_t)\phi_i. \quad (1.8)$$

Often the intensity of the Brownian excitation is constant across regimes and there is no feedback control:

$$dx_t = \sum_i A_i x_t \phi_i\, dt + C\, dw_t. \quad (1.9)$$

We will find this simpler model to be useful in certain tracking applications.

JTS: In some applications, the plant state discontinuity has a particular structure. There is neither rotation nor scaling. The plant state discontinuity is a translation in the form of a difference between mode-specific levels: $\rho(i,l) = \rho_l - \rho_i$. Array these levels as rows of an $s \times n$ matrix $\rho = [\rho_i]$. The base-state dynamic equation of a jump translating system

(JTS) can be written

$$dx_t = \sum_i ((A_i x_t + B_i(u_t - \upsilon\phi_t))\,dt + C_i\,dw_t)\phi_i + (\rho' - \chi)\,d\phi_t. \tag{1.10}$$

If the plant state is a continuous process and there is no control, the JTS-model becomes even simpler:

$$dx_t = \sum_i A_i x_t \phi_i\,dt - \chi\,d\phi_t + C\,dw_t, \tag{1.11}$$

where the model is shown with constant Brownian intensity.

In interpreting the results derived on the basis of (1.7), we should recognize the approximations inherent in the model. If we ignore the drift identified with the feedforward implementation, the intrasojourn base-state dynamics are LGM. This model is easily justified in a region about the set point where higher order deviation variables are negligible. Exactly this kind of linearization procedure is accepted practice in applications involving unimodal plants, and during quiescent periods, Equation (1.5) – the intermodal restriction of (1.7) – is reasonable. If the set point changes, the magnitude of the base-state vector will increase abruptly. The state of a well-regulated plant will move expeditiously toward the new set point. In (1.7) the evolution model uses the dynamics of the successor regime. There are systems for which this concatenation of local models would be inappropriate (e.g., an unstable system moves away from the new set point). We will not pursue this issue further and will accept (1.7) as an adequate for our purposes.

The comprehensive plant state (base, mode) is a combination of continuous and discrete elements. The base-state moves within $\mathbb{R}^n$, and though $\phi_t \in \mathbb{R}^s$, the modal-state has a finite range space. The modal process is usually thought to be exogenous: The path of $\{\Phi_t\}$ is indifferent to $\{x_t\}$. Because it modulates the base-state motion, $\{\Phi_t\}$ is not, however, independent of $\{x_t\}$. With this heterogeneous state space structure, such plants are called *hybrid*. Heterogeneity of various kinds is becoming more common in applications, and the adjective "hybrid" is applied quite broadly. Nevertheless, because it is so descriptive, we will use hybrid to refer to plants and systems with this state space decomposition.

To complete the plant model, the temporal evolution of the modal-state must be quantified. In much of what follows, $\{\phi_t\}$ will be represented by an $\mathcal{F}_t$-Markov process satisfying the stochastic differential equation:

modal-state model

$$d\phi_t = Q'\phi_t\,dt + dm_t \tag{1.12}$$

with initial condition ϕ_0. The $S \times S$ matrix Q is called the modal transition rate matrix: If $i \neq j$, $\mathcal{P}(\phi_{t+dt} = \mathbf{e}_j \,|\, \phi_t = \mathbf{e}_i) = Q_{ij}\,dt$ with $Q_{ii} = -\sum_{l \neq i} Q_{il}$. The off-diagonal elements of the Q-matrix are nonnegative. The diagonal elements are such as to make the row sums of Q equal zero. It is known that the mean sojourn time in state $\phi_t = \mathbf{e}_i$ is $-1/Q_{ii}$, and if $\phi_t = \mathbf{e}_i$, the probability that the next modal transition will be $\mathbf{e}_i \mapsto \mathbf{e}_j$ is $-Q_{ij}/Q_{ii}$. Consequently, Q can be particularized from observations of the modal process. The second term in (1.12) is a purely discontinuous $\mathcal{F}_t$-martingale increment: $E[dm_t \,|\, \mathcal{F}_t] = 0$.

Equation (1.12) can be integrated into (1.7). Note that $\phi_i \mathbf{e}_l' \, d\phi_t = (Q_{il}\,dt + dm_l)\phi_i$. So

$$dx_t = \sum_i ((A_i x_t + B_i(u_t - \upsilon\phi_t))\,dt + C_i\,dw_t)\phi_i + \sum_{i,l} (M(i,l)x_t + (\chi_i - \chi_l) + M(i,l)\chi_i + \rho(i,l))(Q_{il}\,dt + dm_l)\phi_i. \tag{1.13}$$

Though not a particularly appealing relation, (1.13) can be made easier to interpret if we collect some of the terms that have a common influence. Let

$$\begin{aligned} \mathbf{A}_i &= A_i + \sum_l Q_{il} M(i,l), \\ \Theta(i,l) &= \chi_i - \chi_l + M(i,l)\chi_i + \rho(i,l), \\ \boldsymbol{\rho}_{i\cdot} &= \sum_l \Theta(i,l) Q_{il}. \end{aligned} \tag{1.14}$$

In these terms, the base-state model can be written

$$dx_t = \sum_i ((\mathbf{A}_i x_t + B_i(u_t - \upsilon\phi_t))\,dt + C_i\,dw_t)\phi_i + \sum_{i,l} (M(i,l)x_t + \Theta(i,l))\phi_i\,dm_l + \boldsymbol{\rho}'\phi_t\,dt. \tag{1.15}$$

The equation of base-state dynamics has the general appearance of an LGM model but it differs in important particulars. The state matrix, $\mathbf{A}_i$, of $\{x_t\}$ is composed of the intramodal component (A_i) plus a component determined by both the direction of the linear, intermodal discontinuity and its likelihood $(\sum_l Q_{il} M(i,l))$. The control matrix, B_i, is that of the intramodal model. The translational discontinuity in the plant state is reflected in $\boldsymbol{\rho}'\phi_t\,dt$. There is a collection of terms in the drift of $\{x_t\}$ not found in the classical models of control and estimation. The model is highly nonlinear with the modal-state a multiplier throughout.

The increment in $\{x_t\}$ also contains exogenous forcing terms. One is a wideband noise term $(C_i\,dw_t)$ also found in LGM models. The other is neither linear nor Gaussian. The plant state discontinuity term, $\sum_{i,l}(M(i,l)x_t + \Theta(i,l))\phi_i\,dm_l$, is an increment of a purely discontinuous martingale. The coefficient, $(M(i,l)x_t + \Theta(i,l))\phi_i$, contains base- and modal-state products.

The specialized dynamics of an LJS are not changed when the modal process is Markovian because the modal dynamics do not enter the base-state equation. The base-state evolution of the JTS can be written:

$$dx_t = \sum_i ((A_i x_t + B_i(u_t - \upsilon\phi_t) + (\rho' - \chi)Q'\mathbf{e}_i)\,dt + C_i\,dw_t)\phi_i + (\rho' - \chi)\,dm_t. \tag{1.16}$$

Equation (1.16) contains the same types of excitation found in the more comprehensive model, (1.15), but the simpler structure of (1.16) will be reflected in the estimation algorithm; compare $(\rho' - \chi)\,dm_t$ with $\sum_{i,l}(M(i,l)x_t + \Theta(i,l))\phi_i\,dm_l$.

In this book, we will present algorithms for generating (or approximating) $\{\hat{x}_t\}$ and $\{\hat{\phi}_t\}$. The accuracy of the estimates depends upon the quality and kind of sensors available in the application. A model for one kind of sensor is displayed in (1.4). The measurement is time continuous, linear in plant state, and the noise is additive and Gaussian. We will refer to (1.4) as the model of the *plant state sensor* even though $\{y_t\}$ may be generated by a collection of individual devices arrayed in a suite. For example, there may be radars aboard a set of geographically diverse platforms (shipboard, land-based, and air-based) with all tracking the same target. It is this aggregate that is called the plant state sensor. The noise in the observation is determined by both the sensor and the geometry (e.g., range), after linearization if necessary.

When the measurement frequency is too slow to justify using (1.4), the plant state sensor outputs are more accurately viewed as a time-discrete sequence. Suppose observations occur with intersample period T. A linear, time-discrete measurement of the plant state at time $t = kT$ is a direct analogue of (1.4):

plant state measurement: time-discrete

$$y[k] = H\chi[k] + n[k], \tag{1.17}$$

where $\{n[k]\}$ is a Gaussian white noise process with covariance R_x ($R_x > 0$), independent of the exogenous processes in (1.13). As is the case when the measurements are time continuous, if $\{\phi_t\}$ is known, $\{y[k]\}$ can be recast as a measurement of the base-state uncontaminated by the mode: $y[k] - H\chi\phi[k] = Hx[k] + n[k]$, and the *measurement residual* is defined to be the difference between what the output is and what it is predicted to be:

$$r[k] = y[k] - E[y[k]\,|\,\mathcal{G}[k-1]] = H\tilde{x}[k]^{-} + n[k].$$