1

Definitions and Governing Equations

Vorticity plays an important role in fluid dynamics analysis, and in many cases it is advantageous to describe dynamic events in a flow in terms of the evolution of the vorticity field.

The vorticity field ($\boldsymbol{\omega}$) is related to the velocity field ($\mathbf{u}$) of a flow as

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}. \tag{1.0.1}$$

It follows from this definition that vorticity is a solenoidal field:

$$\nabla \cdot \boldsymbol{\omega} = 0. \tag{1.0.2}$$

In a Cartesian coordinate system (x, y, z) this relation yields the following relationships between the velocity componenets (u_x, u_y, u_z) and the vorticity components $(\omega_x, \omega_y, \omega_z)$:

$$\omega_x = \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}, \quad \omega_y = \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}, \quad \omega_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}. \tag{1.0.3}$$

In two dimensions the vorticity field has only one nonzero component (ω_z) orthogonal to the (x, y) plane, thus automatically satisfying solenoidal condition (1.0.2).

The circulation Γ of the vorticity field around a closed curve L, surrounding a surface S with unit normal $\mathbf{n}$ is defined by

$$\Gamma = \int_L \mathbf{u} \cdot d\mathbf{r} = \int_S \boldsymbol{\omega} \cdot \mathbf{n} \, dS, \tag{1.0.4}$$

where $d\mathbf{r}$ denotes an element of the curve.

There are several physical interpretations of the definition of vorticity. We will adopt the point of view that vorticity is a solid-body-like rotation that can be imparted to the elements because of a stress distribution in the fluid. Hence when we consider a vorticity-carrying fluid element, the increment of angular velocity ($d\Omega$) across an infinitesimal distance ($d\mathbf{r}$) over the element is given by

$$d\Omega = \frac{1}{2}\omega \times d\mathbf{r}. \tag{1.0.5}$$

When we can track the translation and deformation of vorticity-carrying fluid elements, because of the kinematics and dynamics of the flow field we are able to obtain a complete description of the flow field. Considering the vorticity-carrying fluid elements as computational elements is the basis of the vortex methods that we analyze in this book. The close link of numerics and physics is the essense of vortex methods, and it is a point of view that will be emphasized throughout this book.

In this introductory chapter we present fundamental definitions and equations relating to the kinematics and the dynamics of the vorticity field. In Section 1.1 we introduce the description of flow phenomena in terms of Eulerian and Lagrangian points of view. Using these two descriptions, we present in Section 1.2 the dynamic laws governing the evolution of the vorticity field in a viscous, incompressible flow field. In Section 1.3 we present Helmholtz's and Kelvin's laws governing the motion of the vorticity field.

1.1. Kinematics of Vorticity

There are two different ways of expressing the behavior of the fluid that may be classified as the Lagrangian and the Eulerian point of view. Their difference lies in the choice of coordinates we wish to use to describe flow phenomena.

1.1.1. Lagrangian Description

When the fluid is viewed as a collection of fluid elements that are freely translating, rotating, and deforming, then we may identify the dependent quantities of the flow field (such as the velocity, temperature, etc.) with these individual fluid elements. In that sense the Lagrangian viewpoint is a natural extension of particle mechanics. To obtain a full description of the flow we need to identify the initial location of the fluid elements and the initial value of the dependent variable. The independent variables are then the initial location of a point ($\mathbf{x}_p^0$) and time (T). By following the trajectories of the collection of fluid elements, we are able to sample at every location in space and instant in time the quantity of interest.

The primary flow quantity in this description is the velocity of the individual fluid elements. The velocity of a fluid element that is residing in an inertial frame of reference at $\mathbf{X}_p$ is expressed as

$$\mathbf{u}_p = \frac{\partial \mathbf{X}_p}{\partial T}. \tag{1.1.1}$$

The acceleration of a fluid particle in a Lagrangian frame is expressed as

$$\mathbf{a}_p = \frac{\partial \mathbf{u}_p}{\partial T}. \tag{1.1.2}$$

The Lagrangian description is ideally suited to describing phenomena in terms of the vorticity of the flow field.

1.1.2. Eulerian Description

In this description of the flow, our observation point is fixed at a certain location $\mathbf{x}$ of the flow field. The flow quantities as they are changing with time t are considered as functions of $\mathbf{x}$. Unlike in Lagrangian methods the location of our observation point remains unchanged by time, and it is the change of the values of the dependent variables at the observation point that describes the flow field.

The Eulerian and the Lagrangian quantities of the flow are related as

$$\mathbf{x} = \mathbf{X}(\mathbf{x}^0, T), \tag{1.1.3}$$

$$t = T. \tag{1.1.4}$$

The Eulerian description of the flow is the most commonly used method to describe flow phenomena in the fluid mechanics literature. In this description, individual fluid elements and their history are not tracked explicitly, but rather it is the global picture of the field that is changing with time that provides us with the description of the flow.

1.1.3. The Material Derivative

The material derivative allows us to relate the Eulerian and the Lagrangian time derivatives of a dependent variable. Let Q be a quantity of the flow expressed in a Lagrangian frame as $Q(\mathbf{x}^0, T)$ and let q be the same quantity expressed in an Eulerian frame, that is, $q(\mathbf{x}, t)$. Then we would have that

$$Q(\mathbf{x}^0, T) \;=\; q[\mathbf{x} = \mathbf{X}(\mathbf{x}^0, T), t]. \tag{1.1.5}$$

So the rate of change of Q with time T may be related to the rate change of q with time t with the chain rule for differentiation as

$$\frac{\partial Q}{\partial T} = \frac{\partial q}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial T} + \frac{\partial q}{\partial t}\frac{\partial t}{\partial T}, \tag{1.1.6}$$

and since we have for the velocity of a fluid particle that $\mathbf{u} = \partial \mathbf{x}/\partial T$ then

$$\frac{\partial Q}{\partial T} = \frac{\partial q}{\partial t} + \mathbf{u} \cdot \frac{\partial q}{\partial \mathbf{x}}. \tag{1.1.7}$$

The first term is the local rate of change of a variable, and the second term is the convective change of the dependent variable. The substantial derivative (i.e., the rate of change of quantity in a Lagrangian frame) is a convenient way of understanding several phenomena in fluid mechanics, and Stokes has given it a special symbol:

$$\frac{\mathrm{D}()}{\mathrm{D}t} = \frac{\partial()}{\partial t} + (\mathbf{u} \cdot \nabla)(). \tag{1.1.8}$$

From the definition of the substantial derivative we may easily see then that

$$\frac{\mathrm{D}\mathbf{x}}{\mathrm{D}t} = \mathbf{u}. \tag{1.1.9}$$

We may also determine the rate of change of a material line element ($d\mathbf{r}$) by using the definition of the substantial derivative as

$$\frac{\mathrm{D}(d\mathbf{r})}{\mathrm{D}t} = d\mathbf{u} = \partial_j \mathbf{u} \, d\mathbf{r}_j = \mathbf{dr} \cdot \nabla \mathbf{u}. \tag{1.1.10}$$

1.1.4. Reynold's Transport Theorem

As an illustrative example of the Lagrangian and the Eulerian descriptions of the flow, we may consider the rate of change of the volume integral of the quantity Q in a material volume $[V(t)]$ with surface $[S(t)]$ having normal $\mathbf{n}$ and velocity $\mathbf{u}$, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} Q \, dV. \tag{1.1.11}$$

Contributions for this rate of change are given by the local rate of change of Q, $\int_{V(t)} \partial Q/\partial t \, dV$, as well as from the motion of the boundary $\int_{S(t)} Q(\mathbf{u} \cdot \mathbf{n}) \, dS$

[note that for small times dt we may write $dV = dS(\mathbf{u}\cdot\mathbf{n})\,dt$] so that we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} Q\,dV = \int_{V(t)} \frac{\partial Q}{\partial t}\,dV + \int_{S(t)} Q(\mathbf{u}\cdot\mathbf{n})\,dS, \tag{1.1.12}$$

By using vector calculus we may write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} Q\,dV = \int_{V(t)} \frac{\partial Q}{\partial t}\,dV + \int_{V(t)} \nabla\cdot(Q\mathbf{u})\,dV, \tag{1.1.13}$$

or by using the expression for the substantial derivative we may write that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} Q\,dV = \int_{V(t)} \frac{\mathrm{D}Q}{\mathrm{D}t}\,dV + \int_{V(t)} Q\nabla\cdot\mathbf{u}\,dV. \tag{1.1.14}$$

which is known as Reynold's transport theorem for the quantity Q.

1.2. Dynamics of Vorticity

The motion of an incompressible Newtonian fluid is governed by the following equations that express the conservation of mass and momentum of fluid in Eulerian and Lagrangian frames [160]. In the Eulerian description we consider the development of the flow field as it is observed at a fixed point P of the domain, while in the Lagrangian description we consider the equations from the point of view of a material fluid element that moves with the local velocity of the flow.

The conservation of mass can be expressed as

Eulerian Description:

$$\frac{\partial\rho}{\partial t} \quad + \quad \nabla\cdot(\rho\mathbf{u}) \quad = \quad 0. \tag{1.2.1}$$

$\frac{\partial\rho}{\partial t}$	$\nabla\cdot(\rho\mathbf{u})$
Rate of accumulation of mass per unit volume at P	Net flow rate of mass out of P per unit volume

Lagrangian Description:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} \quad = \quad -\rho \quad \nabla\cdot\mathbf{u}. \tag{1.2.2}$$

$\frac{\mathrm{D}\rho}{\mathrm{D}t}$	$-\rho$	$\nabla\cdot\mathbf{u}$
Rate of change of the density of a fluid element	Mass per unit volume	Particle-volume expansion rate

The conservation of momentum can be expressed in terms of the velocity ($\mathbf{u}$) and the pressure P of the flow field as

Eulerian Description:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \mu \Delta \mathbf{u}, \tag{1.2.3}$$

Rate of increase of momentum at P	Net flow rate of momentum carried in P by $\rho\mathbf{u}$	Net pressure force	Net viscous force

where μ denotes the dynamic viscosity of the fluid, and $\nu = \mu/\rho$ denotes the kinematic viscosity of the fluid with density ρ.

Lagrangian Description:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla P + \mu \Delta \mathbf{u}. \tag{1.2.4}$$

Acceleration of a fluid particle	Net pressure force	Net viscous force

With definition of vorticity (1.0.1) the momentum equations for an incompressible, Newtonian fluid of uniform density can be expressed in Lagrangian and Eulerian forms as

Eulerian Description:

$$\rho \frac{\partial \boldsymbol{\omega}}{\partial t} + \rho \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \rho \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \mu \Delta \boldsymbol{\omega}. \tag{1.2.5}$$

Rate of increase of vorticity	Net flow rate of vorticity	Vortex stretching	Viscous diffusion

Lagrangian Description:

$$\rho \frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \rho \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \mu \Delta \cdot \boldsymbol{\omega}. \tag{1.2.6}$$

Rate of change of particle vorticity	Rate of deforming vortex lines	Net rate of viscous diffusion

Note that in the velocity–vorticity formulation the pressure of the flow can be recovered from the equation

$$\frac{1}{\rho} \Delta P = -\nabla \cdot \left(\frac{1}{2} |\mathbf{u}|^2 - \mathbf{u} \times \boldsymbol{\omega} \right). \tag{1.2.7}$$

In the case of a viscous, Newtonian flow of a fluid with nonuniform density, rotation can be imparted to the fluid elements because of the baroclinic generation of vorticity. In this case the equation for the vorticity field is

$$\frac{\mathrm{D}(\boldsymbol{\omega}/\rho)}{\mathrm{D}t} = \left(\frac{1}{\rho}\boldsymbol{\omega}\cdot\nabla\right)\mathbf{u} + \nu\Delta\boldsymbol{\omega} + \frac{1}{\rho}\nabla P \times \nabla\frac{1}{\rho}. \tag{1.2.8}$$

1.3. Helmholtz's and Kelvin's Laws for Vorticity Dynamics

In order to characterize the kinematic evolution of the vorticity field it is useful to introduce some geometrical concepts. We consider the vector of the vorticity field and we identify the lines that are tangential to this vector as vortex lines. In turn, a collection of these lines can form vortex surfaces or vector tubes. The motions of fluid elements carrying vorticity obey certain laws that were first outlined by Helmholtz for the inviscid evolution of the vorticity and further extended by Kelvin to include the effects of viscosity.

From the solenoidal condition for the vorticity field, integrating over a volume of fluid with nonzero vorticity, and using the Gauss theorem, we obtain that

$$\int_V \nabla\cdot\boldsymbol{\omega}\,dV = \int_S \boldsymbol{\omega}\cdot\mathbf{n}\,dS = 0, \tag{1.3.1}$$

where V denotes the volume of the fluid encompassed by the surface S. When we consider a vortex tube, Eq. (1.3.1) dictates that the strength of the vortex tube is the same at all cross sections. This is Helmholtz's first theorem. When Eq. (1.3.1) is applied to a vorticity tube with cross sections A_1 and A_2 with respective uniform normal vorticity components $\omega_1 = \boldsymbol{\omega}\cdot\mathbf{n}_1$ and $\omega_2 = \boldsymbol{\omega}\cdot\mathbf{n}_2$ (Fig. 1.1) we obtain that

$$|\omega_1|A_1 = |\omega_2|A_2 = |\Gamma| \tag{1.3.2}$$

independently of the behavior of the vorticity field between the two cross-sections of the vortex tube. Equation (1.3.2) defines the circulation (Γ) of the vortex tube.

When we consider the Lagrangian description of the inviscid evolution of the vorticity field in an incompressible flow (with $\rho = 1$), Eq. (1.2.6) can be expressed as

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \boldsymbol{\omega}\cdot\nabla\mathbf{u}. \tag{1.3.3}$$

Comparing Eqs. (1.3.3) and (1.1.10) for the evolution of material lines,

$$\frac{\mathrm{D}d\mathbf{r}}{\mathrm{D}t} = d\mathbf{r}\cdot\nabla\mathbf{u}, \tag{1.3.4}$$

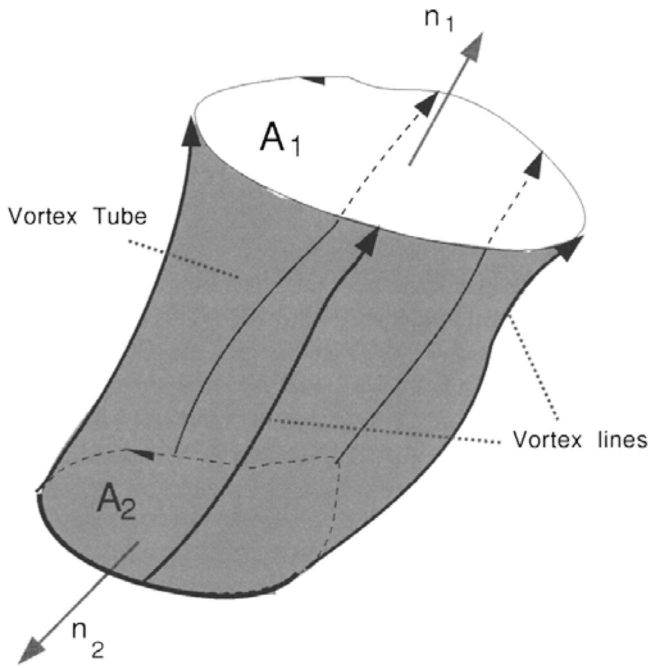

Figure 1.1. Sketch of vortex lines and vortex tube.

we observe that in a circulation-preserving motion the vortex lines are material lines. This is Helmholtz's second theorem for the motion of vorticity elements. As a result of this law, fluid elements that at any time belong to one vortex line, however they may be translated, remain on the vortex line. A result of the first and the second laws is the property of vortex lines and tubes: that no matter how they evolve, they must always form closed curves or they must have their ends in the bounding surface of the fluid.

Kelvin extended the laws of Helmholtz in order to account for the effects of viscosity and at the same time provide a different physical interpretation for the motion of vorticity-carrying fluid elements in terms of the circulation around a closed curve. From the definition of circulation for a line around a cross section of a vortex tube we obtain that

$$\Gamma = \int_L \mathbf{u} \cdot d\mathbf{r}. \tag{1.3.5}$$

Now by using the Lagrangian form of the velocity–pressure formulation for the acceleration of the material particles we obtain

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\int_L \mathbf{u}\cdot d\mathbf{r} \tag{1.3.6}$$

$$= \int_L \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\cdot d\mathbf{r} + \int_L \frac{\mathrm{D}d\mathbf{r}}{\mathrm{D}t}\cdot d\mathbf{u}. \tag{1.3.7}$$

As we are tracking material lines we obtain that

$$\int_L \frac{\mathrm{D}d\mathbf{r}}{\mathrm{D}t}\cdot d\mathbf{u} = \int_L \mathbf{u}\cdot d\mathbf{u} = 0. \tag{1.3.8}$$

Using Eq. (1.3.8) and momentum equation (Eq. 1.2.4), we can express Eq. (1.3.7) as

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \int_L \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\cdot d\mathbf{r} \tag{1.3.9}$$

$$= -\int_L \nabla P\cdot d\mathbf{r} + \nu\int_L \Delta\mathbf{u}\cdot d\mathbf{r}. \tag{1.3.10}$$

Noting that the pressure term integrates to zero, we obtain that

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \nu\int_L (\Delta\mathbf{u})\cdot d\mathbf{r}. \tag{1.3.11}$$

In the case of an inviscid flow, the right-hand side of Eq. (1.3.11) is zero and the circulation of material elements is conserved. This is Kelvin's theorem for the modification of circulation of fluid elements.

In the case of baroclinic flow the circulation around a material line can be modified because of the baroclinic generation of vorticity, and Kelvin's theorem is modified as

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \nu\int_L (\Delta\mathbf{u})\cdot d\mathbf{r} + \int \frac{1}{\rho^2}\nabla\rho\times\nabla P\cdot\mathbf{n}\,dS. \tag{1.3.12}$$

Note that the second term on the right-hand side is an integral over the area encompassed by the material curve. Equation (1.3.12) is known as Bjerken's theorem.

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2

Vortex Methods for Two-Dimensional Flows

The simulation of phenomena governed by the two-dimensional Euler equations are the first and simplest example in which vortex methods have been successfully used. The reason can be found in Kelvin's theorem, which states that the circulation in material – or Lagrangian – elements is conserved. Mathematically, this comes from the conservative form of the vorticity equation. Following markers – or particles – where the local circulation is concentrated is thus rather natural. At the same time, the nonlinear coupling in the equations resulting from the velocity evaluation immediately poses the problem of the mollification of the particles into blobs and of the overlapping of the blobs, which soon was realized to be a central issue in vortex methods.

The two-dimensional case thus encompasses some of the most important features of vortex methods. We first introduce in Section 2.1 the properties of vortex methods by considering the classical problem of the evolution of a vortex sheet. We present in particular the results obtained by Krasny in 1986 [129, 130] that demonstrated the capabilities of vortex methods and played an important role in the modern developments of the method. We then give in Sections 2.2 to 2.4 a more conventional exposition of vortex methods and of the ingredients needed for their implementation: choice of cutoff functions, initialization procedures, and treatment of periodic boundary conditions. Section 2.6 is devoted to the convergence analysis of the method and to a review of its conservation properties.

2.1. An Introduction to Two-Dimensional Vortex Methods: Vortex Sheet Computations

The origin of vortex methods may be traced back to the 1920s and 1930s in the works of Prager (1928) [163] and Rosenhead (1931) [172, 173]. They